RUSSIAN ROULETTE AND PARTICLE SPLITTING

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INTRODUCTION

Two situations are encountered in particle transport simulations:

1. In a multiplying medium, a particle such as a neutron a cosmic ray particle or a photon may generate a large number of secondary particles.
2. The followed particle remains within the considered region without escape, leading to an excessively long scattering chain.

In either case a method for terminating the particle chains is needed, while maintaining the process unbiased is needed. Russian Roulette and particle splitting are two complementary importance-sampling methods which can achieve that goal.

RUSSIAN ROULETTE

In the spirit of Tolstoi's War and Peace, the Russian Roulette game is played among people who have consumed a substantial amount of spirits, historically vodka, and are in the unfortunate possession of a six-shooter. To display their courage in the royal salons instead of on the battle-field, or maybe to show their disdain for life, a single bullet is loaded into the gun, the barrel flipped, more drinks are consumed to completely loose touch with reality, then the players take turns aiming the gun at their temples and pressing the trigger. At each trial, the probability of the person pulling the trigger proving his stupidity beyond a reasonable doubt and thus deserving to shorten his life and promptly meeting his creator is 1/6. He needs to repeat his brave feat for only six times to face a probability of one of blowing off his brain and ruining the expensive carpet under his feet.

Luckily in Monte Carlo this game is applied to fictitious particles in particle transport simulations. Consider a particle whose statistical weight is:

$$w_{old}.$$

The Russian Roulette random variable can be described as:

$$\left\{ \begin{array}{ll} p & (1-p) \\ kill\ \ particle & keep\ \ particle \\ w_{new} = 0 & w_{new} = w_{old}\dfrac{1}{(1-p)} \end{array} \right\} \tag{1}$$

That is, with probability p, the particle is killed and with probability (1-p), the particle survives, but its statistical weight is adjusted to the new value:

$$w_{new} = w_{old} \frac{1}{(1-p)} \tag{2}$$

The important consideration is that the expected weight of the particle is preserved at the same that it had before the collision. In fact the expected value of the statistical weight is:

$$\begin{aligned} E(w_{new}) &= p.0 + (1-p).w_{old} \cdot \frac{1}{(1-p)} \\ &= w_{old} \end{aligned} \tag{3}$$

As an example , let us consider a choice of the kill probability as:

p=0.5,

which kills the particles 50 percent of the time, and lets them survive 50 percent of the time.

The particles that are killed are eliminated from the simulation and new source particles are sampled. Those that survive will now be assigned a statistical weight given by Eqn, 2 as:

$$\begin{aligned} w_{new} &= w_{old} \frac{1}{(1-0.5)} \\ &= 2w_{old} \end{aligned} \tag{4}$$

That is the surviving particles will have their statistical weight doubled.

In another case if we choose:

p=0.9,

particles will be killed 90 percent of the time while those surviving the process will be assigned a new weight given by:

$$\begin{aligned} w_{new} &= w_{old} \frac{1}{(1-0.9)} \\ &= 10w_{old} \end{aligned} \tag{5}$$

The process is repeated with different values of the kill probability p until the number of particles is reduced to a manageable size. It may even be necessary to use it in some simulations to end up the generated otherwise infinite chains. Thus it provides a convenient way to terminate particle histories, while preserving their original statistical weight.

PARTICLE SPLITTING

Conversely to the Russian Roulette technique, the sample size can be increased, while avoiding a bias by using the Splitting method which.

Consider a particle of statistical weight:

$$w_{old}.$$

It can be split to any number n of new identical particles each with statistical weight w_i such that:

$$w_i = \frac{w_{old}}{n} \;,\; i = 1, 2, \dots, n \tag{6}$$

The expected value of the new particles weights is their sum:

$$\begin{aligned} E\left(\sum_{i=1}^{n} w_i\right) &= E(w_1 + w_2 + \dots w_n) \\ &= n.\frac{w_{old}}{n} \\ &= w_{old} \end{aligned} \tag{7}$$

which preserves the initially split particle statistical weight w_{old}.

By such a weighting method, one controls the total number of tracks, and the relative numbers of tracks in various regions of phase space, which is equivalent to a form of Importance Sampling.

Ideally, the number of paths would be proportional to their contribution to the final result, with avoidance of those paths that do not contribute to the sought answer in the spirit of any well thought Importance Sampling method.

COUPLED RUSSIAN ROULETTE AND PARTICLE SPLITTING

When Russian Roulette and Particle Splitting are combined, particles will tend to have nearly equal weights, which is advantageous in reducing the variance in the computed quantity of interest.

Consider a medium subdivided into boundaries:

$$r_1, r_2, \dots, r_i, \dots r_n$$

each of them assigned an importance:

$$I_1, I_2, \ldots, I_i, \ldots I_n,$$

as shown in Fig. 1.

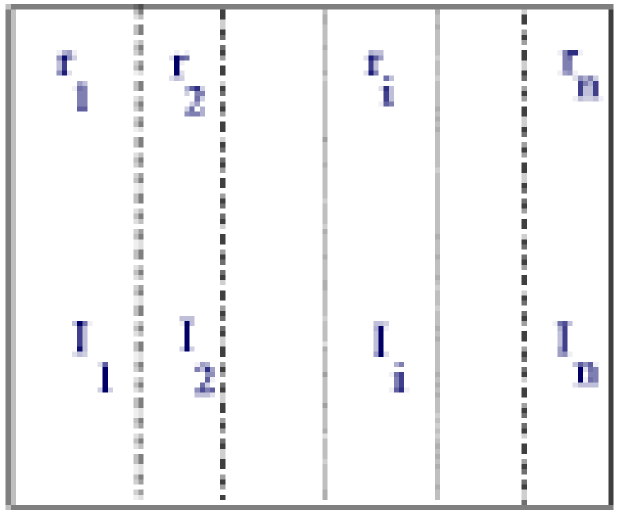

Figure 1. Regions assignment of the importance values.

When a particle enters a new region (i+1) from region i, we define the relative importance of the regions as:

$$\nu = \frac{I_{i+1}}{I_i} \tag{8}$$

is calculated. If the new region that is entered has greater importance than the previous one:

$$\nu = \frac{I_{i+1}}{I_i} > 1$$

the particle is split into ν identical particles, each carrying a statistical weight:

$$w_{new} = w_{old} \frac{1}{\nu} \tag{9}$$

If on the other hand:

$$\nu' = \frac{I_{i+1}}{I_i} < 1,$$

Russian Roulette is played, with the particle killed with a probability:

$$p_{kill} = 1 - \nu' \tag{10}$$

The particle survives with a probability:

$$p_{survival} = \nu' \tag{11}$$

The surviving particle is kept in the simulation with a new statistical weight:

$$\begin{aligned} w_{new} &= \frac{w_{old}}{p_{survival}} \\ &= w_{old} \cdot \frac{1}{\nu'} \end{aligned} \tag{12}$$

If the regions have equal importance:

$$\nu'' = \frac{I_{i+1}}{I_i} = 1,$$

neither Russian Roulette nor Splitting are played.

Notice that like in the case of Russian Roulette, the expected value of the statistical weight for the particle is preserved:

$$\begin{aligned} E(w_{new}) &= w_{old} \cdot \frac{1}{\nu'} . p_{survival} + 0.p_{kill} \\ &= w_{old} \cdot \frac{1}{\nu'} . \nu' + 0.(1 - \nu') \\ &= w_{old} \end{aligned} \tag{13}$$

As an example of the application of the methodology, if the relative importance of a region relative to the previous region is:

$$\nu = \frac{I_{i+1}}{I_i} = 2 > 1,$$

the incoming particle is split into:

$$\nu = 2 \quad \text{particles},$$

each carrying a statistical weight:

$$w^1{}_{new} = w_{old} \cdot \frac{1}{2}$$
$$w^2{}_{new} = w_{old} \cdot \frac{1}{2}$$

If the relative importance of a region is less than the previous region:

$$\nu' = \frac{I_{i+1}}{I_i} = 0.5 < 1,$$

Russian Roulette is applied and the particle is killed with a probability:

$$p_{kill} = 1 - 0.5 = 0.5,$$

the survival probability being:

$$p_{survival} = 0.5$$

The surviving particle weight is:

$$w_{new} = w_{old} \cdot \frac{1}{0.5} = 2w_{old}$$

EXERCISE

1. For the example on the application of the combined Russian Roulette and Particle Splitting techniques, prove that the expected value of the particle statistical weight is preserved.