SAMPLING SPECIAL DISTRIBUTIONS

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INTRODUCTION

Monte Carlo simulations require the sampling of various distributions at different steps in the simulation process. There exist a multitude of approximate algorithms that have been ingeniously devised to sample some specified functions. In other cases special sources need to be sampled for starting up the solution or to account for the boundary or initial conditions.

Sampling these distributions adequately depends upon correctly identifying the underlying probability density functions, deducing the corresponding cumulative distribution functions, and next inverting them to sample the source, the boundary or initial condition. Several geometric sources are considered to demonstrate the approach.

SAMPLING ISOTROPIC SOURCES

We consider the sampling of an isotropic spherically symmetric source of particles or photons. In this case each element of solid angle receives the same contribution from the isotropic source as shown in Fig. 1. This means we have to sample the probability density function;

$$p(\theta,\phi)d\theta d\phi = \frac{d\Omega}{4\pi} = \frac{dS}{4\pi r^2} = \frac{r^2 \sin\theta d\theta d\phi}{4\pi r^2}$$

$$= p(\theta)d\theta.p(\phi)d\phi = \frac{\sin\theta d\theta}{2} \cdot \frac{d\phi}{2\pi} \tag{1}$$

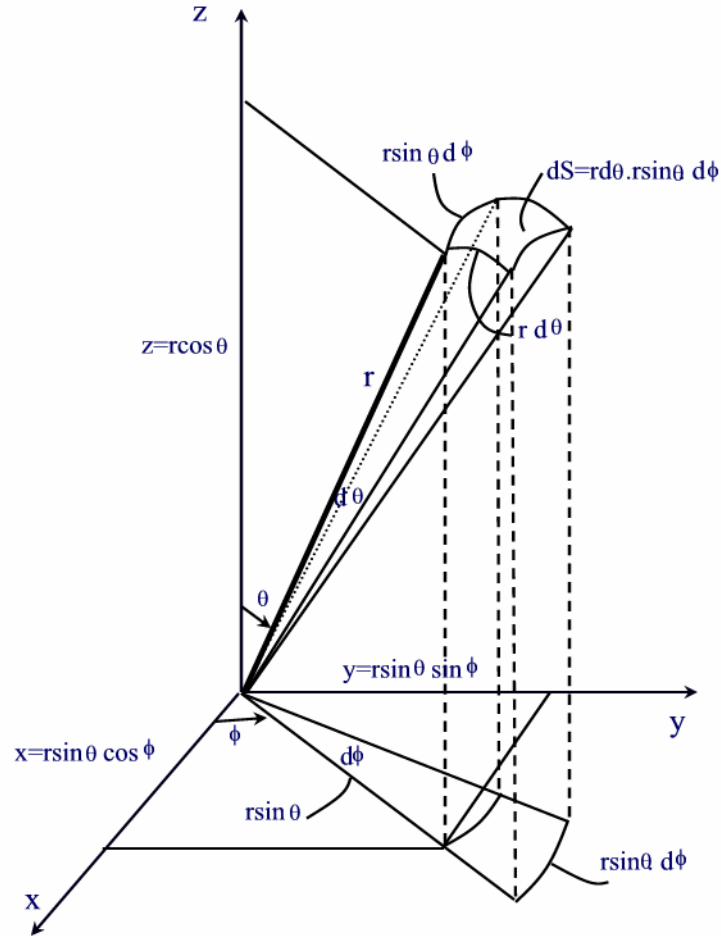where: the element of solid angle $d\Omega = \dfrac{dS}{r^2}$.

Here the probability density function is separable into two probability density functions:

$$p(\theta)d\theta = \frac{\sin\theta d\theta}{2}, \quad \int_0^\pi \frac{\sin\theta}{2}d\theta = 1 \tag{2}$$

$$p(\phi)d\phi = \frac{d\phi}{2\pi}, \quad \frac{1}{2\pi}\int_0^{2\pi} d\phi = 1 \tag{3}$$

These probability density functions represent two independent random variables, and can be sampled separately. Setting:

$$C(\phi) = \int_0^{\phi} \frac{d\phi}{2\pi} = \rho_1, \qquad (4)$$

Fig. 1: Sampling an isotropic spherically symmetric source.

we can sample the azimuthal angle from:

$$\phi(\rho) = 2\pi\rho_1 \qquad (5)$$

Setting:

$$C(\theta) = \int_0^{\theta} \frac{\sin\theta}{2} d\theta = \rho_2,$$

$$1 - \cos\theta = 2\rho_2,$$

$$\cos\theta = 1 - 2\rho_2.$$

or the polar angle can be sampled as:

$$\theta = \cos^{-1}(1 - 2\rho_2) \tag{6}$$

Now that the polar and azimuthal angles have been sampled, we can determine the direction cosines u, v, and w for the particles or photons emanating from the isotropic source according to the simple relationships:

$$u = \frac{x}{r} = \frac{r\sin\theta\cos\phi}{r} = \sin\theta\cos\phi$$

$$v = \frac{y}{r} = \frac{r\sin\theta\sin\phi}{r} = \sin\theta\sin\phi \tag{7}$$

$$w = \frac{z}{r} = \frac{r\cos\theta}{r} = \cos\theta$$

SAMPLING THE NORMAL OR GAUSSIAN DISTRIBUTION

Invoking the Central Limit Theorem one can generate a Normal of Gaussian variable in terms of a sequence of pseudo random variables uniformly distributed over the unit interval. For such a random variable the mean value is:

$$\mu = \frac{\int_0^1 xf(x)dx}{\int_0^1 f(x)dx} = \frac{\int_0^1 xdx}{\int_0^1 dx} = [\frac{x^2}{2}]_0^1 = \frac{1}{2}, \tag{8}$$

and the variance is:

$$\sigma^2 = \frac{\int_0^1 x^2 f(x)dx}{\int_0^1 f(x)dx} - \mu^2 = \frac{\int_0^1 x^2 dx}{\int_0^1 dx} - \mu^2 = [\frac{x^3}{3}]_0^1 - \left(\frac{1}{2}\right)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}, \tag{9}$$

Thus its standard deviation is:

$$\sigma = \frac{1}{\sqrt{12}} \tag{10}$$

According to the Central Limit Theorem, the random variable:

$$\xi = \frac{\sum\limits_{i=1}^{n} x_i - n\mu}{\sqrt{n}\,\sigma} \tag{11}$$

converges asymptotically with n to a Normal distribution with mean μ and variance σ^2: $N[\mu,\sigma]$:

$$N(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]}, \quad -\infty < x \le \infty \tag{12}$$

For the sampling of a uniform random distribution, we can substitute from Eqs. 9 an 11 for the values of the mean and standard deviation as:

$$\xi = \frac{\sum\limits_{i=1}^{n} r_i - \frac{n}{2}}{\sqrt{\frac{n}{12}}} \tag{13}$$

As a good approximation, we can take n=12 in Eqn. 13 to yield:

$$\xi = \sum_{i=1}^{12} r_i - 6 \tag{14}$$

which yields a very simple algorithm to generate a Normal or Gaussian variate:

1. Generate a sequence of 12 random numbers r_i, i=1,2, ..., 12, over the unit interval.
2. Let: $\xi = \sum\limits_{i=1}^{12} r_i - 6$,
3. Report a sample: $\xi' = \mu + \xi\sigma$ from $N[\mu,\sigma^2]$.

Sometimes the 90 percent error spread factor is used instead of the standard deviation as a parameter:

$$\xi' = \mu + \xi(1.645\,\sigma).$$

A procedure that generates a Normal random variable by the above-mentioned algorithm is shown in Fig. 2. It uses calls to a subroutine to reconstruct a Normal variable that is displayed in Fig. 3.

```fortran
!          normal_gaussian_distribution.f90
!          Sampling the Normal or Gaussian distribution
!          M. Ragheb, Univ. of Illinois
           program normal_gaussian_distribution
           dimension e(100), freq(100)
           integer :: trials = 100000
           real x, e, freq, score
           real :: xmean =10.0
           real :: std_dev = 3.0
           score = 1.0
!  Initialize frequency distribution
           do i=1,20
                   freq(i)=0.0
           end do
!          Initialize bins
           do i=1,20
                   xi=i
                   e(i)=xi
           end do
!          open output file
           open(44, file = 'random_out')
!  Sample distribution
           do i= 1, trials
!  Sample x coordinate
                   call normal(xmean,std_dev,x,rr)
!  Construct frequency distribution
                   if(x.LE.e(1))then
                    freq(1)=freq(1)+score
                   end if
                   do j=1,20
                           if((x.GT.e(j)).AND.(x.LE.e(j+1)))then
                           freq(j+1)=freq(j+1)+score
                           end if
                   end do
           end do
!          Normalize frequency distribution and construct pdf
           do i=1,20
                   freq(i)=freq(i)/trials
           end do
!          Write results to output file
           do i=1,20
                   write (44,100)e(i),freq(i)
                   write(*,*)e(i),freq(i)
           end do
100        format (2e14.8)
           end

           subroutine normal(xmean,std_dev,x,rr)
!          This subroutine generates a Normal or Gaussian random variable
!          with a mean value xmean and a standard deviation std_dev
!          mean value= xmean
!          standard deviation= std_dev
!          returned sampled point= x
!          9o% error spread= s = 1.645*standard deviation
           n=12
           xn=n
!          s=1.645*std_dev
```
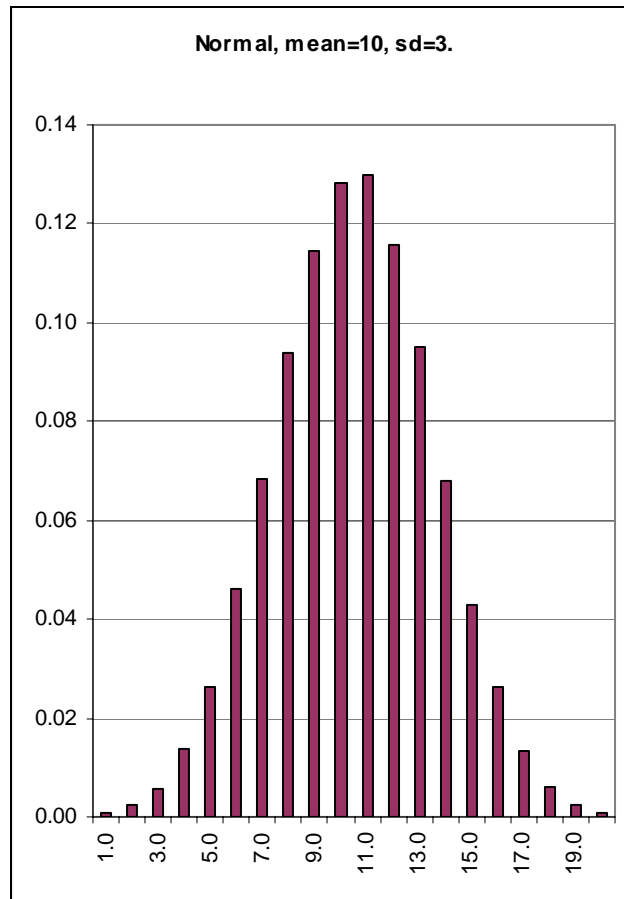
```
s=std_dev
const = sqrt (xn/12.0)
sum = 0.0
do i=1,n
          call random(rr)
          sum = sum + rr
end do
v=(sum-(xn/2.0))/const
x=xmean + v*s
return
end
```

Fig. 2: Procedure to sample and display the Normal or Gaussian distribution.

Fig. 3: Sampled Normal or Gaussian distribution, N=100,000 trials.

SAMPLING THE LOGNORMAL DISTRIBUTION

A distribution related to the Normal distribution is the Lognormal or Logarithmico-Normal distribution. If X is from $N(\mu, \sigma^2)$, then Y=e^X has the Lognormal distribution with a pdf:

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma y} e^{\left[-\frac{(\ln y - \mu)^2}{2\sigma^2}\right]}, \ 0 < y \le \infty \qquad (15)$$

An algorithm based on the Central Limit Theorem can thus be proposed to generate the Lognormal distribution:

1. Generate a sequence of 12 random numbers r_i, i=1,2, …, 12, over the unit interval.

2. Let: $\xi = \sum_{i=1}^{12} r_i - 6$,

3. Generate a sample: $X = \mu + \xi\sigma$ from $N[\mu, \sigma^2]$.

4. Report Y=e^X as a sample from the Lognormal distribution.

A procedure using this algorithm is shown in Fig. 4, where a subroutine is called to generate the samples, and the ensuing Lognormal distribution is shown in Fig. 5.

```
!          lognormal_distribution.f90
!          Sampling the lolgnormal distribution
!          M. Ragheb, Univ. of Illinois
           program lognormal_distribution
           dimension e(101), freq(101)
           integer :: trials = 1000000
           real x, e, freq, score
           real :: xmean =20.0
           real :: std_dev = 3.0
           score = 1.0
!          Initialize frequency distribution
           do i=1,100
                   freq(i)=0.0
           end do
!          Initialize bins
           do i=1,100
                   xi=i
                   e(i)=xi
           end do
!          open output file
           open(44, file = 'random_out')
!          Sample distribution
           do i= 1, trials
!          Sample x coordinate
                   call lognormal(xmean,std_dev,x,rr)
!          Construct frequency distribution
                   if(x.LE.e(1))then
                    freq(1)=freq(1)+score
                   end if
                   do j=1,100
                           if((x.GT.e(j)).AND.(x.LE.e(j+1)))then
                           freq(j+1)=freq(j+1)+score
                           end if
                   end do
           end do
!          Normalize frequency distribution and construct pdf
           do i=1,100
                   freq(i)=freq(i)/trials
```

```
        end do
!       Write results to output file
        do i=1,100
                write (44,100)e(i),freq(i)
                write(*,*)e(i),freq(i)
        end do
100     format (2e14.8)
        end


        subroutine lognormal(xmean,std_dev,x,rr)
!       This subroutine generates a Lognormal random variable
!       from a normal distribution with zero mean and unit standard
!       deviation
!       mean value= xmean
!       standard deviation= std_dev
!       returned sampled point= x
!       9o% error spread = s = 1.645*standard deviation
        n=12
        xn=n
        xxmean= alog (xmean)
        sstd_dev = alog (std_dev)
!       s=1.645*std_dev
        s=sstd_dev
        const = sqrt (xn/12.0)
        sum = 0.0
        do i=1,n
                call random(rr)
                sum = sum + rr
        end do
        v=(sum-(xn/2.0))/const
        xx=xxmean + v*s
        x=exp(xx)
        return
        end
```

Fig. 4: Procedure for the sampling and display of the lognormal distribution.

SAMPLING LINE SOURCES

Sampling different source geometrical configurations is encountered in many problems such as shielding against radiation, and heat transport problems. It depends upon correctly identifying the underlying probability density function, deducing the corresponding cumulative distribution function and then inverting it to sample the source. Several geometries are encountered in practice. We consider the most important ones of them to demonstrate the approach to sampling them.

The probability density function for a line source of length H along the z coordinate axis can be written as:

$$pdf : f(z)dz = \frac{dz}{\int_0^H dz} = \frac{dz}{H} \qquad (16)$$

Notice the normalization in the denominator. The geometry used in the sampling is shown in Fig. 6.
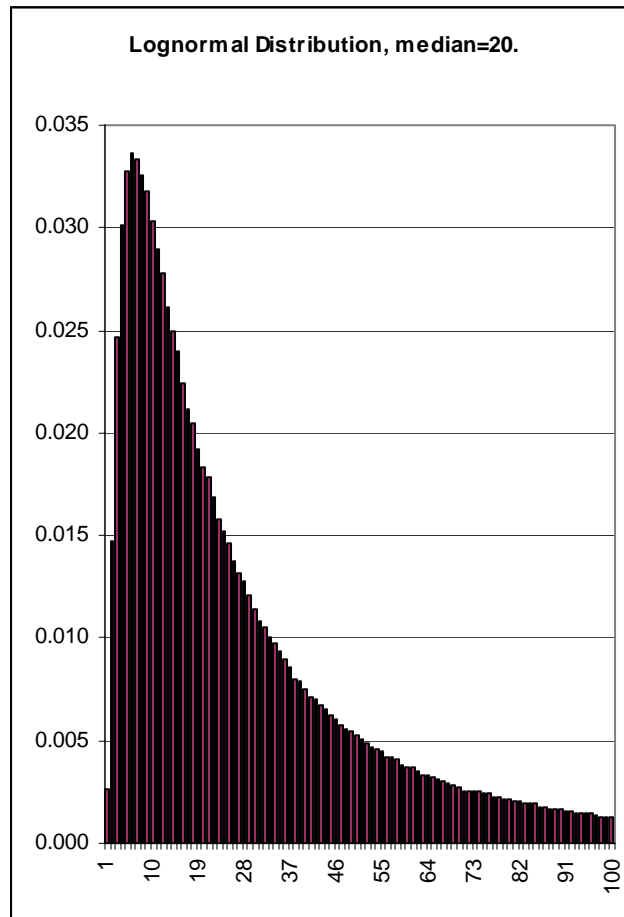
Fig. 5: A sampled lognormal distribution with median=20.
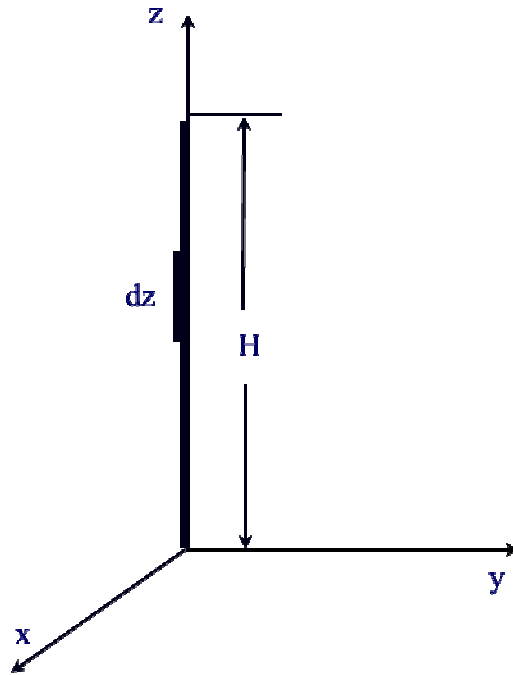
The cumulative distribution function is:

$$cdf : C(z) = \frac{\int_0^z dz}{H} = \frac{z}{H} = \rho \qquad (17)$$

Equating the cdf to a pseudo random number uniformly distributed over the unit interval allows us to sample the positions along the line source as:
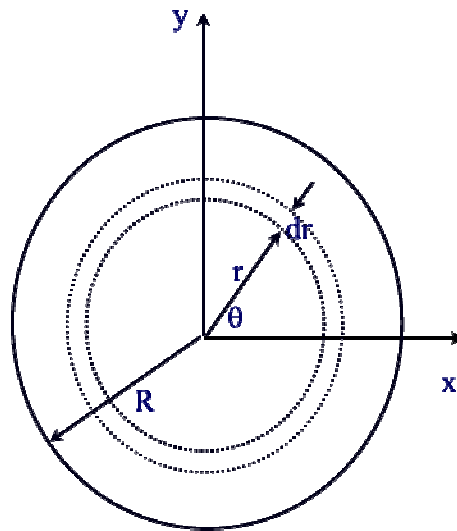
$$z = H\rho \qquad (18)$$

SAMPLING A DISK SOURCE

Considering a disk centered at the origin with radius R, we can deduce its pdf and the cdf, as shown in Fig. 7, or the radius r we get:

Fig. 6: Line Source geometry.

Fig. 7: Disk Source geometry

```
!          disk_source.f90
!          Sampling sources uniformly distributed on a disk of radius R
!   pdf = f(r) = 2*pi*r / pi*R**2
```

```
!           cdf = C(r) = r**2/R**2
!  pdf = f(phi) = 1/(2*pi)
!           cdf       = C(phi) = phi/(2*pi)
!           M. Ragheb, Univ. of Illinois
!           program disk_source
!           Variables declaration
            real :: radius  = 1.0
            real :: pi = 3.14159
            integer :: trials = 1000
            real r, phi, x, y
!           Open output file
            open(44, file = 'random_out')
!  Begin sampling points
            do i= 1, trials
!           Sample radius on disk
                   call random(rr)
                   r= radius * sqrt (rr)
!           Alternate source sampling
!                  r=radius * rr
!           Sample azimuthal angle
                   call random(rr)
                   phi = 2.0*pi*rr
!           Calculate x and y coordinates
                   x = r * cos (phi)
                   y = r * sin (phi)
!  Write sampled points to output file
                   write (44,100) x, y
                   write(*,*) x, y
100                format (2e14.8)
            end do
            end
```

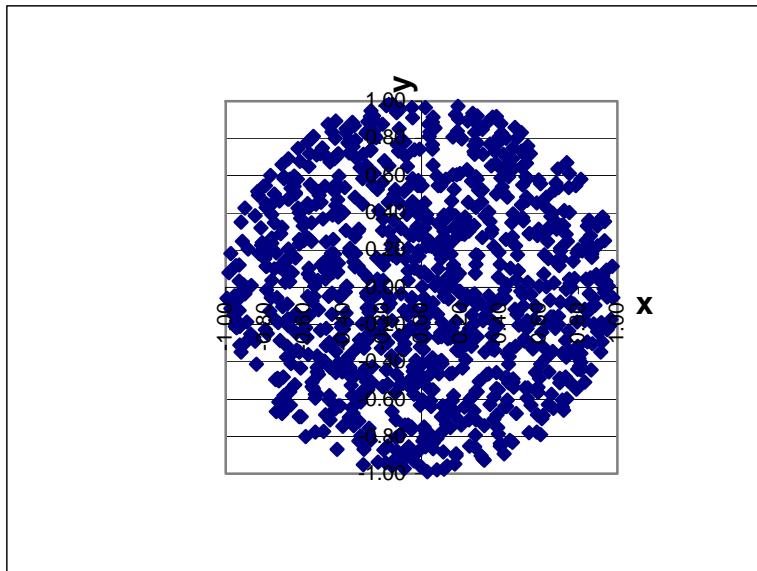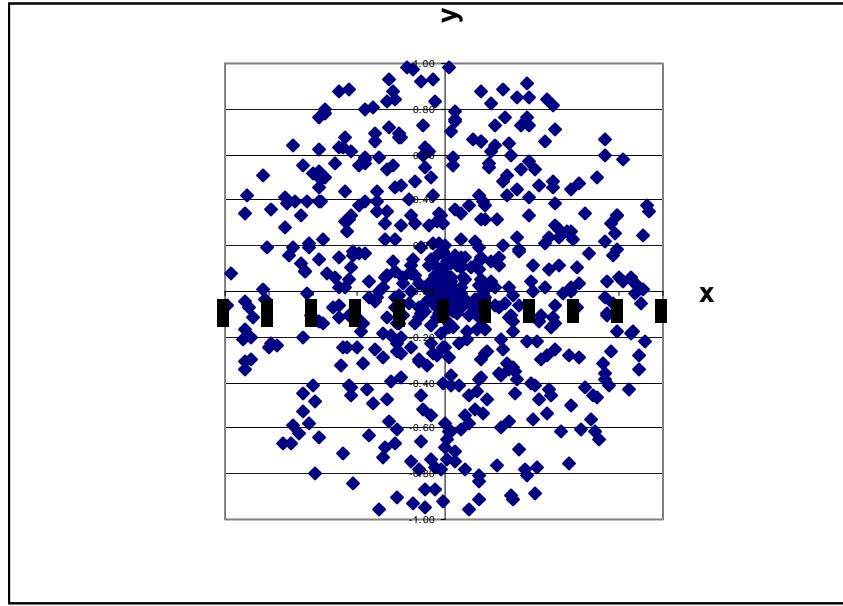**Fig. 8: Disk source sampling procedure.**

Fig. 9: Uniformly distributed source on a unit radius disk. N=1,000.

Fig. 10: Incorrectly sampled distributed source on a unit radius disk. N=1,000.

$$pdf : f(r)dr = \frac{dA}{\int dA} = \frac{2\pi r dr}{\int_0^R 2\pi r dr} = \frac{2\pi r dr}{\pi R^2} \tag{19}$$

$$cdf : C(r) = \frac{\int_0^r 2\pi r dr}{\pi R^2} = \rho_1 \tag{20}$$

Which can be inverted to sample the radius r as:

$$r = R\rho_1^{1/2} \tag{21}$$

For the angle θ we can write the pdf and cdf:

$$pdf : f(\theta)d\theta = \frac{d\theta}{\int_0^{2\pi} d\theta} = \frac{d\theta}{2\pi} \tag{22}$$

$$cdf : C(\theta) = \frac{\int_0^\theta d\theta}{2\pi} = \frac{\theta}{2\pi} = \rho_2 \qquad (23)$$

Inverting the cdf we get the sampled angle as;

$$\theta = 2\pi\rho_2 \qquad (24)$$

Notice that we needed here to generate two pseudo random numbers for the radius and angular variables. The x and y coordinates for a point on the disk can thus be deduced as:

$$x = r\cos\theta = R\rho_1^{1/2}\cos(2\pi\rho_2) \qquad (25)$$

$$y = rs\,os\,\theta = R\rho_1^{1/2}\sin(2\pi\rho_2) \qquad (26)$$

We could have opted to sample the angle θ over the interval [-π,+π]. In this case we choose:

$$\theta' = \pi(2\rho_2 - 1) \qquad (27)$$

since the cdf is this case:

$$cdf : C(\theta') = \frac{\int_{-\pi}^\theta d\theta'}{2\pi} = \frac{\theta' + \pi}{2\pi} = \rho_2 \qquad (28)$$

Figure 8 shows a procedure for sampling a disk source. The uniform source sampled with 1,000 points is shown in Fig. 9. If one had chosen the source as proportional to the pseudo random number rather than its square root, a nonuniform source would have ensued as shown in Fig. 10.

SAMPLING A RING SOURCE

This can be considered as a special case of the disk source with pdf and cdf for the radial parameter r as:

$$pdf : f(r)dr = \frac{dA}{\int dA} = \frac{2\pi r dr}{\int_{R_1}^{R_2} 2\pi r dr} = \frac{2\pi r dr}{\pi(R_2^2 - R_1^2)} \qquad (29)$$

where R1 and R2 are the inner and outer radii of the ring respectively, as shown in Fig. 11.

$$cdf : C(r) = \frac{\int_{R_1}^{r} 2\pi r \, dr}{\pi(R_2^2 - R_1^2)} = \frac{\pi(r^2 - R_1^2)}{\pi(R_2^2 - R_1^2)} = \rho \qquad (30)$$

By inversion we can sample the radius r on the ring as:

$$r^2 = R_1^2 + (R_2^2 - R_1^2)\rho$$
$$r = [R_1^2 + (R_2^2 - R_1^2)\rho]^{1/2} \qquad (31)$$

The sampling of the angular variable would be the same as in the case of the disk source.

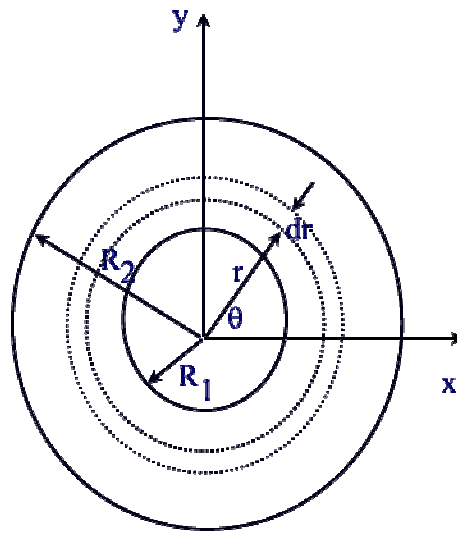**Fig. 11: Ring Source geometry.**

```
!          ring_source.f90
!          Sampling sources uniformly distributed on a disk of radius R
!  pdf = f(r) = 2*pi*r / pi*(R2**2-R1**2)
!          cdf = C(r) = (r**2-R1**2)/(R2**2-R1**2)
!  pdf = f(phi) = 1/(2*pi)
!          cdf      = C(phi) = phi/(2*pi)
!          M. Ragheb, Univ. of Illinois.
           program ring_source
!          Variables declaration
           real :: radius2  = 1.0
           real :: radius1  = 0.5
           real :: pi = 3.14159
           integer :: trials = 500
           real r, phi, x, y
!          Open output file
           open(44, file = 'random_out')
           r1=radius1*radius1
```

```
        r2=radius2*radius2
        dr=r2-r1
!  Begin sampling points
        do i= 1, trials
!        Sample radius on disk
                call random(rr)
                r= sqrt (r1+dr*rr)
!        Sample azimuthal angle
                call random(rr)
                phi = 2.0*pi*rr
!        Calculate x and y coordinates
                x = r * cos (phi)
                y = r * sin (phi)
!  Write sampled points to output file
                write (44,100) x, y
                write(*,*) x, y
100             format (2e14.8)
        end do
        end
```

Fig. 12: Procedure for sampling a uniform ring source.
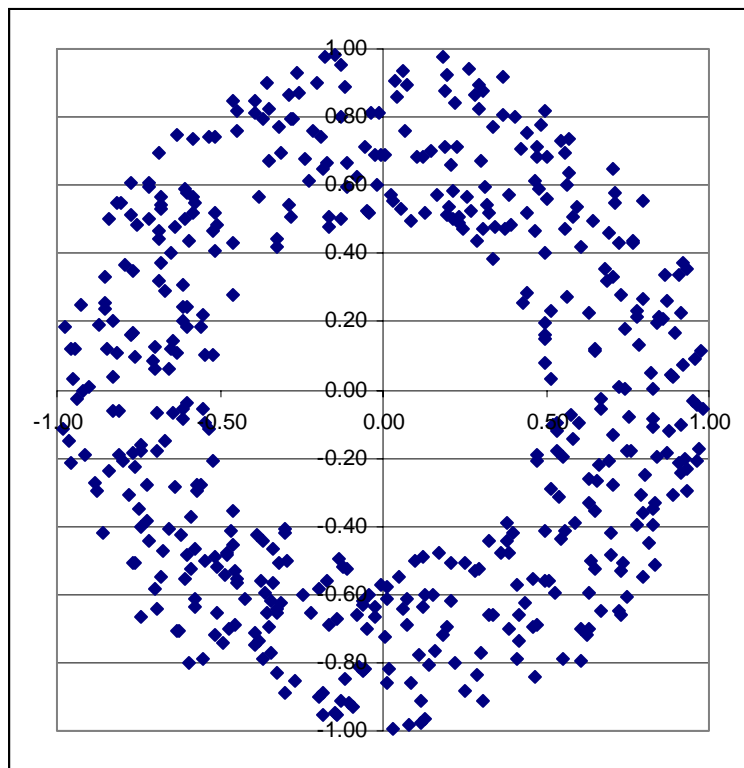
Fig. 13: Sampled points of a uniform ring source.

SAMPLING A CYLINDRICAL SOURCE

The procedure to sample points uniformly distributed over the volume of a cylindrical source such a fuel element in a nuclear reactor, a spent fuel casket, or the reactor core itself, is a combination of the sampling of the line source and the disk source. In this case the cylindrical probability density function is:

$$pdf : p(r,\theta,z)drd\theta dz = \frac{2\pi rd\theta dzdr}{\int\limits_0^{2\pi}\int\limits_0^H\int\limits_0^R 2\pi rd\theta dzdr}$$

$$= \frac{2\pi rdr}{\int\limits_0^R 2\pi rdr} \cdot \frac{dz}{\int\limits_0^H dz} \cdot \frac{d\theta}{\int\limits_0^{2\pi} d\theta}$$

(32)

yields sampled points uniformly distributed over the cylinder volume using three pseudo-random numbers as:

$$r = R\rho_1^{1/2}$$
$$\theta = 2\pi\rho_2$$
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = H\rho_3$$

(33)

```fortran
!        cylinder_source.f90
!        Sampling sources uniformly distributed in a cylinder of radius R
!        and height H
!   pdf = f(r) = 2*pi*r / pi*R**2  cdf = C(r) = r**2/R**2
!   pdf = f(phi) = 1/(2*pi)        cdf  = C(phi) = phi/(2*pi)
!        pdf = f(z) = 1/H          cdf = C(z) = z/H
!        M. Ragheb, Univ. of Illinois.
!        program cylinder_source
!        Variables declaration
         real :: radius  = 1.0
         real :: height  = 1.0
         real :: pi = 3.14159
         integer :: trials = 1000
         real r, phi, x, y, z
!        Open output file
         open(44, file = 'random_out')
!  Begin sampling points
         do i= 1, trials
!        Sample radius on disk
                call random(rr)
                r= radius * sqrt (rr)
!        Alternate radius sampling
!                r= radius * rr
!        Sample azimuthal angle
                call random(rr)
                phi = 2.0*pi*rr
```

```
!          Sample height
                 call random(rr)
                 z= height * rr
!          Calculate x and y coordinates
                 x = r * cos (phi)
                 y = r * sin (phi)
!  Write sampled points to output file
                 write (44,100) x, y, z
                 write(*,*) x, y, z
100              format (3e14.8)
        end do
        end
```
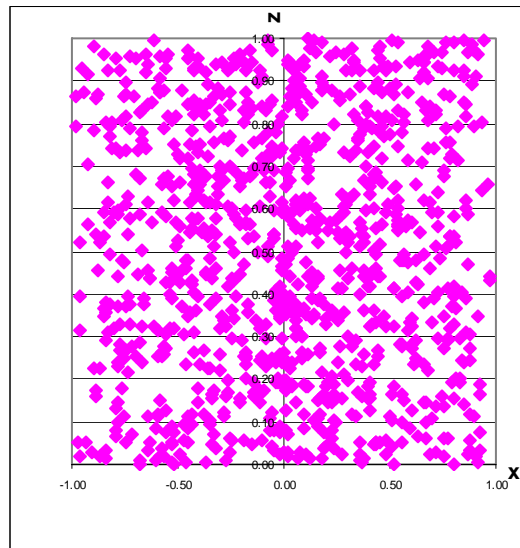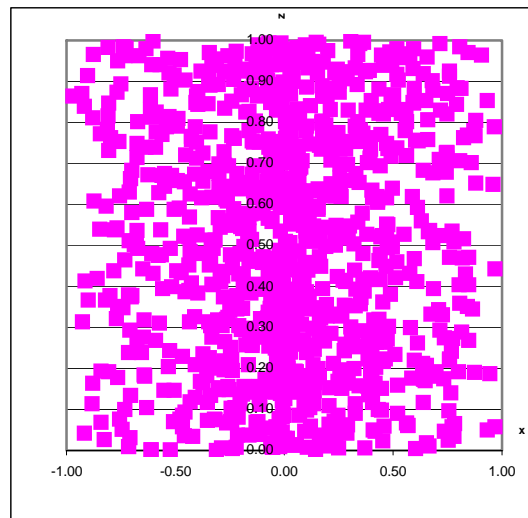
Fig. 14: Procedure for sampling cylindrical source.

Fig. 15: Sampled points for uniform cylindrical source. N=1,000.

Fig. 16: Sampled points for nonuniform cylindrical source. N=1,000.

Figure 14 shows a procedure for sampling a cylindrical source. The uniform source sampled with 1,000 points is shown in Fig. 15. If one had chosen the source as proportional to the pseudo random number rather than its square root, a nonuniform source would have ensued as shown in Fig. 16.

SAMPLING A SPHERICAL SOURCE

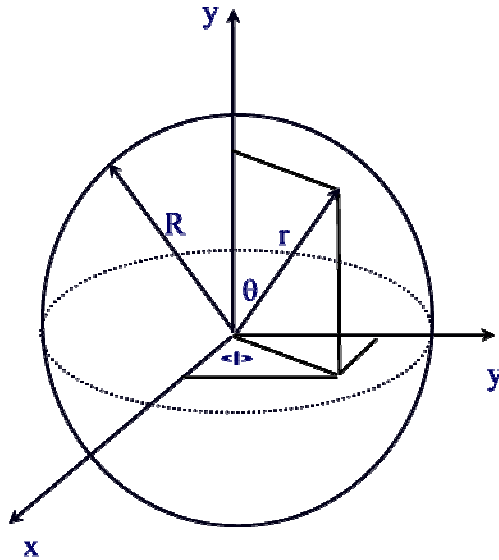For a sphere of radius R we can write for the pdf and cdf in the radial direction:

$$pdf : f(r)dr = \frac{dV}{\int dV} = \frac{4\pi r^2 dr}{\int_0^R 4\pi r^2 dr} = \frac{4\pi r^2 dr}{\frac{4}{3}\pi R^3} \tag{34}$$

$$cdf : C(r) = \frac{\int_o^r 4\pi r^2 dr}{\frac{4}{3}\pi R^3} = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3} = \rho_1 \tag{35}$$

Inverting the cdf yields for the sampled radius r:

$$r = R\rho_1^{1/3} \tag{36}$$

Notice the different exponent in the case of the spherical case (1/3) compared with the case of the cylindrical geometry (1/2).

Fig. 17: Spherical source geometry.

For the polar angle θ and the azimuthal angle φ, we adopt the approach of sampling an isotropic spherically symmetrical source. In this case each element of solid angle receives the same contribution from the source. Hence we need to sample the pdf;

$$p(\theta,\phi)d\theta d\phi = \frac{d\Omega}{4\pi} = \frac{1}{4\pi}\frac{dS}{r^2} = \frac{rd\theta.r\sin\theta d\phi}{4\pi r^2}$$

$$= p(\theta).p(\phi) = \frac{\sin\theta d\theta}{2}.\frac{d\phi}{2\pi}$$

For the polar angle:

$$pdf: \ f(\theta)d\theta = \frac{\sin\theta d\theta}{2}, \quad \int_0^\pi \frac{\sin\theta}{2}d\theta = 1$$

$$cdf: \ C(\theta) = \int_0^\theta \frac{\sin\theta}{2}d\theta$$

$$= \frac{1-\cos\theta}{2} = \rho_2$$

$$\mu = \cos\theta = (1-2\rho_2)$$

$$\theta = \cos^{-1}(1-2\rho_2)$$

For the azimuthal angle:

$$pdf: \ f(\phi)d\phi = \frac{d\phi}{\int_0^{2\pi} d\phi} = \frac{d\phi}{2\pi}$$

$$cdf: \ C(\phi) = \int_0^\phi \frac{d\phi}{2\pi} = \frac{\phi}{2\pi} = \rho_3$$

$$\phi = 2\pi\rho_3$$

Thus the sampled coordinate points in a uniform spherical source would be:

$$x = r\sin\theta\cos\phi = R\rho_1^{1/3}(1-\mu^2)^{1/2}\cos(2\pi\rho_3)$$

$$y = r\sin\theta\sin\phi = R\rho_1^{1/3}(1-\mu^2)^{1/2}\sin(2\pi\rho_3) \qquad (37)$$

$$z = r\cos\theta = R\rho_1^{1/3}\mu,$$

$$where: \mu = (1-2\rho_2)$$

Since:

$$\cos^2\theta + \sin^2\theta = \mu^2 + \sin^2\theta = 1$$

$$\sin\theta = (1-\mu^2)^{\frac{1}{2}}$$

SAMPLING A HEMISPHERICAL SOURCE

This is a special case of the spherical source where the polar angle is as shown in Fig. 18:

$$\theta \in [0,\frac{\pi}{2}].$$

Thus:

$$pdf: \ f(\theta)d\theta = \sin\theta, \ \int_0^{\pi/2} \sin\theta d\theta = 1$$

Since:

$$p(\theta,\phi)d\theta d\phi = \frac{d\Omega}{2\pi} = \frac{1}{2\pi}\frac{dS}{r^2} = \frac{rd\theta.r\sin\theta d\phi}{2\pi r^2}$$

$$= p(\theta)d\theta.p(\phi)d\phi = \sin\theta d\theta.\frac{d\phi}{2\pi}$$
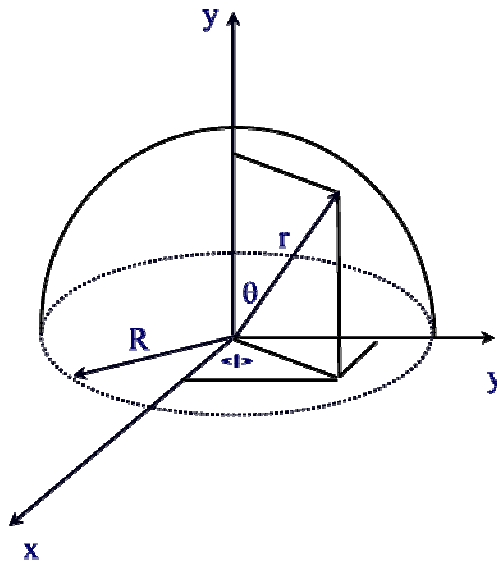
Fig. 18: Hemispherical source geometry

The cdf is:

$$cdf : C(\theta) = \int_0^\theta \sin\theta d\theta$$

$$= 1 - \cos\theta = \rho_2 \tag{38}$$

$$\mu = \cos\theta = (1 - \rho_2) \approx \rho_2$$

$$\theta = \cos^{-1}(1 - \rho_2) \approx \cos^{-1}\rho_2$$

since $(1 - \rho_2)$ is distributed in the same manner as ρ_2.

The sampling for the azimuthal angle proceeds in the same way as for the spherical source.

SAMPLING A SPHERICAL SHELL

The pdf and cdf for the radial position are in this case:

$$pdf : f(r)dr = \frac{dV}{\int dV} = \frac{4\pi r^2 dr}{\int_{R_1}^{R_2} 4\pi r^2 dr} = \frac{4\pi r^2 dr}{\frac{4}{3}\pi(R_2^3 - R_1^3)}$$

$$cdf : C(r) = \frac{\int_{R_1}^r 4\pi r^2 dr}{\frac{4}{3}\pi(R_2^3 - R_1^3)} = \frac{\frac{4}{3}\pi(r^3 - R_1^3)}{\frac{4}{3}\pi(R_2^3 - R_1^3)} = \rho_1 \tag{39}$$

$$r = \left[R_1^3 + (R_2^3 - R_1^3)\rho_1 \right]^{1/3}$$

The sampling of the polar and azimuthal angles is similar to the case of the hemisphere, yielding the samples points:

$$x = r\sin\theta\cos\phi = \left[R_1^3 + (R_2^3 - R_1^3)\rho_1 \right]^{1/3} (1 - \mu^2)^{1/2} \cos(2\pi\rho_3)$$

$$y = r\sin\theta\sin\phi = \left[R_1^3 + (R_2^3 - R_1^3)\rho_1 \right]^{1/3} (1 - \mu^2)^{1/2} \sin(2\pi\rho_3) \tag{40}$$

$$z = r\cos\theta = \left[R_1^3 + (R_2^3 - R_1^3)\rho_1 \right]^{1/3} \mu,$$
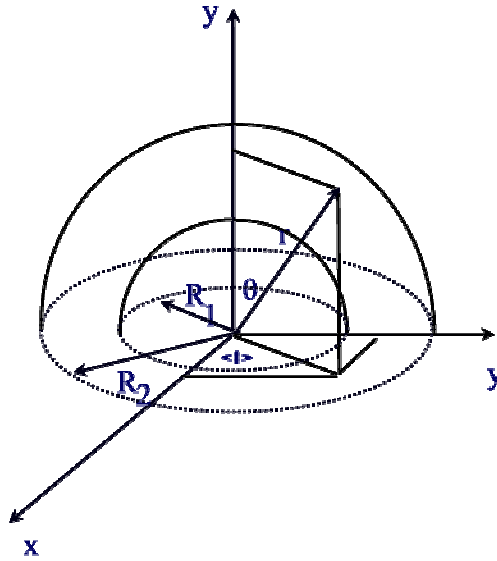
$$where : \mu = (1 - 2\rho_2)$$

SAMPLING A HEMISPHERICAL SHELL

This is a special case of the spherical shell as shown in Fig.19 where:

$$f(\theta)d\theta = \sin\theta d\theta$$

$$C(\theta) = \int_0^\theta \sin\theta d\theta$$

$$= 1 - \cos\theta = \rho_2$$

$$\mu = \cos\theta = (1 - \rho_2) \approx \rho_2$$

And the sampled points become:

$$x = \left[R_1^3 + (R_2^3 - R_1^3)\rho_1\right]^{\frac{1}{3}} (1-\mu^2)^{\frac{1}{2}} \cos(2\pi\rho_3)$$

$$y = \left[R_1^3 + (R_2^3 - R_1^3)\rho_1\right]^{\frac{1}{3}} (1-\mu^2)^{\frac{1}{2}} \sin(2\pi\rho_3) \qquad (41)$$

$$z = \left[R_1^3 + (R_2^3 - R_1^3)\rho_1\right]^{\frac{1}{3}} \mu$$

Fig. 19: Sampling hemispherical shell geometry.

SAMPLING NON-UNIFORM SOURCES

In most situations one encounters cases where the source distribution is not uniform. For instance the permeability in a porous medium, the conductivity in a conductor or the space charge in a plasma may depend on the coordinates, and would be non-uniform. In flow problems, in particular, one encounters the need to set a random radius in a uniform axi-symmetric flow. As an example we consider a spherical source case with a variate such as the probability of a radius r is proportional to r. A volumetric density of source particles describing this situation is given by:

$$d(r) = k.r \tag{42}$$

where k is a constant.

The pdf in this case is modified from Eqn. 35 as:

$$pdf : f(r)dr = \frac{4\pi r^2 (kr)dr}{\int_0^R 4\pi r^2 (kr)dr} = \frac{4\pi k r^3 dr}{\int_0^R 4\pi k r^3 dr} = \frac{4\pi r^3 dr}{\pi R^4} \tag{43}$$

and the cdf is:

$$cdf : C(r) = \frac{\int_o^r 4\pi r^3 dr}{\pi R^4} = \frac{\pi r^4}{\pi R^4} = \rho \tag{44}$$

Thus the sampled radius is given by:

$$r = R\rho^{1/4} \tag{45}$$

The polar and azimuthal angles can be sampled in a way similar to the case for a uniform spherical source.

SAMPLING DISTRIBUTIONS WHICH ARE NOT PROBABILITY DENSITY FUNCTIONS

Consider a random variable that is defined over the interval:

$$\xi \in [3,5]$$

that is distributed proportional to x^3 in this region. The function x^3 can be converted to a probability density function with proper normalization. In this case the pdf is:

$$f(x)dx = \frac{x^3 dx}{\int_3^5 x^3 dx} = \frac{x^3 dx}{[x^4/4]_3^5} = \frac{x^3 dx}{\frac{(5^4 - 3^4)}{4}}$$

The cumulative distribution function is constructed and equated to pseudo random number ρ:

$$C(x) = \frac{\int_{3}^{x} x^3 dx}{\int_{3}^{5} x^3 dx} = \frac{\frac{1}{4}(x^4 - 3^4)}{\frac{(5^4 - 3^4)}{4}} = \frac{(x^4 - 3^4)}{(5^4 - 3^4)} = \rho$$

Inverting the cdf, we get the sampling function:

$$x = [3^4 + (5^4 - 3^4)\rho]^{\frac{1}{4}}. \tag{46}$$

SURFACE SOURCE EMITTING A COSINE DISTRIBUTION

This situation occurs in dosimetry and shielding studies as well as in fluid dynamics when a particle reservoir is used as a boundary condition. For a total surface source strength S_0 [particles/(cm^2.sec)], the source differential angular distribution is given by:

$$S(\theta) = S_0 \frac{\cos\theta}{2\pi} [particles /(\sec.cm^2.steradian)] \tag{47}$$

since:

$$d\Omega = \frac{dS}{r^2} = \frac{r^2 \sin\theta d\theta d\phi}{r^2} = \sin\theta d\theta d\phi$$

The un-normalized pdf for sampling such a source is given by:

$$p(\theta,\phi)d\theta d\phi = S_0 \cos\theta. \frac{d\Omega}{2\pi}$$
$$= S_0 \cos\theta. \frac{\sin\theta d\theta d\phi}{2\pi}$$
$$= f(\theta)d\theta.f(\phi)d\phi$$
$$= S_0 \sin\theta \cos\theta d\theta. \frac{d\phi}{2\pi}$$

The pdf for the polar angle needs to be normalized as discussed in the last section:

$$pdf : f(\theta)d\theta = \frac{S_0 \sin\theta\cos\theta d\theta}{S_0 \int_0^{\pi/2} \sin\theta\cos\theta d\theta}$$

$$= \frac{-\cos\theta d(\cos\theta)}{-\int_0^{\pi/2} \cos\theta d(\cos\theta)}$$

$$= \frac{2\cos\theta d(\cos\theta)}{[\cos^2\theta]_0^{\pi/2}}$$

$$= -2\cos\theta d(\cos\theta)$$

The cdf becomes:

$$cdf : C(\theta) = -\int_0^\theta 2\cos\theta d(\cos\theta)$$

$$= \int_\theta^0 2\cos\theta d(\cos\theta)$$

$$= 1 - \cos^2\theta = \rho$$

Inverting the cdf:

$$\cos^2\theta = 1 - \rho \approx \rho$$
$$\cos\theta = \rho^{1/2} \qquad\qquad (48)$$
$$\theta = \cos^{-1}(\rho^{1/2})$$

SAMPLING SPATIAL DISTRIBUTIONS

We consider the case of sampling the flux or power distribution in a rectangular parallelepiped reactor core with side lengths a, b, and c in the x, y, and z directions respectively, as shown in Fig. 20. Sampling the neutron flux may be needed for shielding calculations.

The neutron source is proportional to the flux distribution:

$$\phi(x, y, z) = \phi_0 \cos(\frac{\pi x}{a})\cos(\frac{\pi y}{b})\cos(\frac{\pi z}{c}) \qquad\qquad (49)$$

If one would sample a uniform source distribution the corresponding pdf would be:

$$f(x, y, z)dxdydz = \frac{dV}{\int dV}$$

$$= \frac{dxdydz}{\int_{-a/2}^{+a/2} dx \int_{-b/2}^{+b/2} dy \int_{-c/2}^{+c/2} dz}$$

$$= \frac{dxdydz}{abc}$$
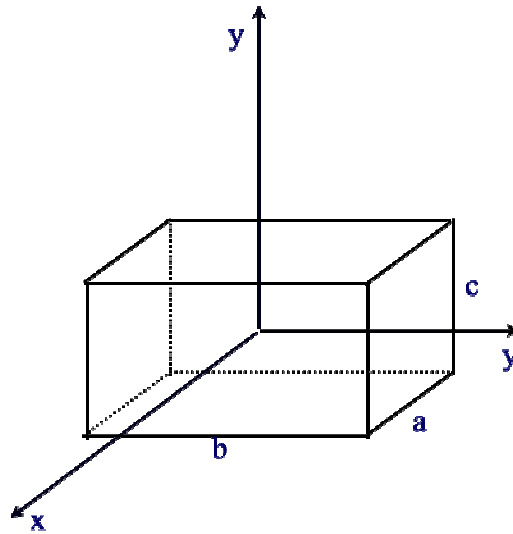
Fig. 20: Flux distribution in a rectangular parallelepiped reactor.

This is separable in the x, y, and z directions as follows:

$$f(x, y, z)dxdydz = f(x)dx.f(y)dy.f(z)dz$$

$$= \frac{dx}{a}\frac{dy}{b}\frac{dz}{c}$$

The corresponding cdfs and sampled points become:

$$C(x) = \frac{\int_{-a/2}^{x} dx}{a} = \frac{x + \frac{a}{2}}{a} = \rho_1 \Rightarrow x = a\rho_1 - \frac{a}{2}$$

$$C(y) = \frac{\int_{-b/2}^{y} dy}{b} = \frac{y + \frac{b}{2}}{b} = \rho_2 \Rightarrow y = b\rho_2 - \frac{b}{2} \tag{50}$$

$$C(z) = \frac{\int_{-c/2}^{z} dz}{c} = \frac{z + \frac{c}{2}}{c} = \rho_3 \Rightarrow z = c\rho_3 - \frac{c}{2}$$

This result can be generalized if the flux distribution of Eqn. 49 is considered. In this case the pdf can be written as:

$$f(x, y, z)dxdydz = \frac{\phi(x, y, z)dV}{\int \phi(x, y, z)dV}$$

$$= \frac{\phi_0 \cos(\frac{\pi x}{a})\cos(\frac{\pi y}{b})\cos(\frac{\pi z}{c})dxdydz}{\phi_0 \int_{-a/2}^{+a/2} \cos(\frac{\pi x}{a})dx \int_{-b/2}^{+b/2} \cos(\frac{\pi y}{b})dy \int_{-c/2}^{+c/2} \cos(\frac{\pi z}{c})dz}$$

Since:

$$\int_{-a/2}^{+a/2} \cos\frac{\pi x}{a}dx = \frac{a}{\pi}[\sin\frac{\pi x}{a}]_{-a/2}^{+a/2} = \frac{2a}{\pi}$$

we can write:

$$f(x, y, z)dxdydz = f(x)dx.f(y)dy.f(z)dz$$

$$= \frac{\pi}{2a}\cos(\frac{\pi x}{a})dx.\frac{\pi}{2b}\cos(\frac{\pi y}{b})dy.\frac{\pi}{2c}\cos(\frac{\pi z}{c})dz$$

The corresponding cdfs are:

$$C(x) = \frac{\pi}{2a} \int_{-a/2}^{x} \cos(\frac{\pi x}{a})dx = \frac{1}{2}[\sin(\frac{\pi x}{a})+1] = \rho_1$$

$$C(y) = \frac{\pi}{2b} \int_{-b/2}^{y} \cos(\frac{\pi y}{b})dy = \frac{1}{2}[\sin(\frac{\pi y}{b})+1] = \rho_2$$

$$C(x) = \frac{\pi}{2c} \int_{-c/2}^{z} \cos(\frac{\pi z}{c})dz = \frac{1}{2}[\sin(\frac{\pi z}{c})+1] = \rho_3$$

The sampled points representing the neutron source distribution will be:

$$x = \frac{a}{\pi}\sin^{-1}(2\rho_1 - 1)$$

$$y = \frac{b}{\pi}\sin^{-1}(2\rho_2 - 1) \tag{51}$$

$$z = \frac{c}{\pi}\sin^{-1}(2\rho_3 - 1)$$

EXERCISES

1.   Write a procedure to sample the Maxwell-Boltzman distribution for velocities and the distribution for energies at room temperature. Make sure that the form you use is normalized. Graphically estimate the most probable energy and velocity, as well as the average velocity and energy.

2.   Write a procedure to simulate the process of radioactive decay of a radioactive isotope with decay constant λ, whose probability density function is:

$$pdf : p(t) = \lambda e^{-\lambda t}$$

$$\lambda = \frac{\ln 2}{T_{1/2}} = \frac{0.6931}{T_{1/2}}$$

where: $T_{1/2}$ is the half life of the isotope.
Simulate the decay of tritium whose half-life is 12.34 [years], and display the frequency distribution that the procedure generates.

3.   In the sampling procedure for the Normal distribution based upon the Central Limit Theorem, the number of uniformly distributed random variables is taken as 12. Investigate the effect on the generated distribution if a smaller number, say 5 is used, or a larger number, say 20 is used.

4.   Modify the procedure to sample a uniform cylindrical source to sample a cylindrical shell of height h, and radii R_1 and R_2.