FLUID MECHANICS EQUATIONS

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INTRODUCTION

The early part of the 18th-century saw the burgeoning of the field of theoretical fluid mechanics pioneered by Leonhard Euler and the father and son Johann and Daniel Bernoulli.

We introduce the equations of continuity and conservation of momentum of fluid flow and the Navier-Stokes and Euler equations as the fundamental relations governing deterministic or mechanistic reactor safety analysis.

THE MASS CONSERVATION OR CONTINUITY EQUATION

The continuity equation of fluid mechanics expresses the notion that mass cannot be created nor destroyed or that mass is conserved. It relates the flow field variables at a point of the flow in terms of the fluid density and the fluid velocity vector, and is given by:

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \bar{V}) = 0 \qquad (1)$$

We consider the vector identity resembling the chain rule of differentiation:

$$\nabla.(\rho \bar{V}) \equiv \rho \nabla.\bar{V} + \bar{V}.\nabla \rho \qquad (2)$$

where the divergence operator is noted to act on a vector quantity, and the gradient operator acts on a scalar quantity.

This allows us to rewrite the continuity equation as:

$$\frac{\partial \rho}{\partial t} + \bar{V}.\nabla \rho + \rho \nabla.\bar{V} = 0 \qquad (3)$$

SUBSTANTIAL DERIVATIVE

We can use the substantial derivative:

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t}_{\substack{Local \\ Derivative}} + (\bar{V}.\nabla)_{\substack{Convetive \\ Derivative}} \qquad (4)$$

where the partial time derivative is called the local derivative and the dot product term is called the convective derivative.

In terms of the substantial derivative the continuity equation can be expressed as:

$$\frac{D\rho}{Dt} + \rho\nabla.\bar{V} = 0 \qquad (5)$$

MOMENTUM CONSERVATION OR EQUATION OF MOTION

Newton's second law is frequently written in terms of acceleration and a force vectors as:

$$\bar{F} = m\bar{a} \qquad (6)$$

A more general form describes the force vector as the rate of change of the momentum vector as:

$$\bar{F} = \frac{d}{dt}(m\bar{V}) \qquad (7)$$

Its general form is written in term of volume integrals and a surface integral over an arbitrary control volume v as a function of the pressure p, the body forces f and the viscous forces $F_{viscous}$:

$$\oiiint_{\mathrm{v}} \frac{\partial(\rho\bar{V})}{\partial t} d\mathrm{v} + \oiint_{S} (\rho\bar{V}.dS)\bar{V} = -\oiiint_{\mathrm{v}} \nabla p d\mathrm{v} + \oiiint_{\mathrm{v}} \rho \bar{f} d\mathrm{v} + \oiiint_{\mathrm{v}} \bar{F}_{viscous} d\mathrm{v} \qquad (8)$$

where the velocity vector is:

$$\bar{V} = u\hat{x} + v\hat{y} + w\hat{z} \qquad (9)$$

The Cartesian coordinates x, y and z components of the continuity equation are:

$$\begin{aligned}
&\oiiint_{\mathrm{v}} \frac{\partial(\rho u)}{\partial t} d\mathrm{v} + \oiint_{S} (\rho\bar{V}.dS)u = -\oiiint_{\mathrm{v}} \frac{\partial p}{\partial x} d\mathrm{v} + \oiiint_{\mathrm{v}} \rho f_x d\mathrm{v} + \oiiint_{\mathrm{v}} (F_x)_{viscous} d\mathrm{v} \\
&\oiiint_{\mathrm{v}} \frac{\partial(\rho v)}{\partial t} d\mathrm{v} + \oiint_{S} (\rho\bar{V}.dS)v = -\oiiint_{\mathrm{v}} \frac{\partial p}{\partial y} d\mathrm{v} + \oiiint_{v} \rho f_y d\mathrm{v} + \oiiint_{\mathrm{v}} (F_y)_{viscous} d\mathrm{v} \qquad (10) \\
&\oiiint_{\mathrm{v}} \frac{\partial(\rho w)}{\partial t} d\mathrm{v} + \oiint_{S} (\rho\bar{V}.dS)w = -\oiiint_{\mathrm{v}} \frac{\partial p}{\partial z} d\mathrm{v} + \oiiint_{v} \rho f_z d\mathrm{v} + \oiiint_{\mathrm{v}} (F_z)_{viscous} d\mathrm{v}
\end{aligned}$$

In this equation the product:

$$(\rho\bar{V}.dS) \qquad (11)$$

is a scalar and has no components.

NAVIER STOKES EQUATIONS

By using the divergence or Gauss's theorem the surface integral can be turned into a volume integral:

$$\begin{aligned}
\oiint_S (\rho\bar{V}.dS)u &= \oiint_S (\rho u\bar{V}).dS = \oiiint_{\mathrm{v}} \nabla.(\rho u\bar{V})d\mathrm{v} \\
\oiint_S (\rho\bar{V}.dS)v &= \oiint_S (\rho v\bar{V}).dS = \oiiint_{\mathrm{v}} \nabla.(\rho v\bar{V})d\mathrm{v} \qquad (12) \\
\oiint_S (\rho\bar{V}.dS)w &= \oiint_S (\rho w\bar{V}).dS = \oiiint_{\mathrm{v}} \nabla.(\rho w\bar{V})d\mathrm{v}
\end{aligned}$$

The volume integrals over an arbitrary volume now yield:

$$\begin{aligned}
\frac{\partial(\rho u)}{\partial t} + \nabla.(\rho u\bar{V}) &= -\frac{\partial p}{\partial x} + \rho f_x + (F_x)_{viscous} \\
\frac{\partial(\rho v)}{\partial t} + \nabla.(\rho v\bar{V}) &= -\frac{\partial p}{\partial y} + \rho f_y + (F_y)_{viscous} \qquad (13) \\
\frac{\partial(\rho w)}{\partial t} + \nabla.(\rho w\bar{V}) &= -\frac{\partial p}{\partial z} + \rho f_z + (F_z)_{viscous}
\end{aligned}$$

These are known as the Navier-Stokes equations. They apply to the unsteady, three dimensional flow of any fluid, compressible or incompressible, viscous or inviscid.

In terms of the substantial derivative, the Navier-Stokes equations can be expressed as:

$$\begin{aligned}
\rho\frac{Du}{Dt} &= -\frac{\partial p}{\partial x} + \rho f_x + (F_x)_{viscous} \\
\rho\frac{Dv}{Dt} &= -\frac{\partial p}{\partial y} + \rho f_y + (F_y)_{viscous} \qquad (14) \\
\rho\frac{Dw}{Dt} &= -\frac{\partial p}{\partial z} + \rho f_z + (F_z)_{viscous}
\end{aligned}$$

EULER EQUATIONS

For a steady state flow the time partial derivatives vanish. For inviscid flow the viscous terms are equal to zero. In the absence of body forces the f_x, f_y, anf f_z terms disappear. The Euler equations result as:

$$\nabla.(\rho u\bar{V}) = -\frac{\partial p}{\partial x}$$
$$\nabla.(\rho v\bar{V}) = -\frac{\partial p}{\partial y} \qquad (15)$$
$$\nabla.(\rho w\bar{V}) = -\frac{\partial p}{\partial z}$$

CONSERVATION OF MOMENTUM FOR INVISCID COMPRESSIBLE FLOW

For an inviscid flow without body forces, the momentum conservation equations of fluid mechanics are:

$$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x}$$
$$\rho\frac{Dv}{Dt} = -\frac{\partial p}{\partial y} \qquad (16)$$
$$\rho\frac{Dw}{Dt} = -\frac{\partial p}{\partial z}$$

These equations can also be written as:

$$\frac{\partial(\rho u)}{\partial t} + \bar{V}.\nabla(\rho u) = -\frac{\partial p}{\partial x}$$
$$\frac{\partial(\rho v)}{\partial t} + \bar{V}.\nabla(\rho v) = -\frac{\partial p}{\partial y} \qquad (17)$$
$$\frac{\partial(\rho w)}{\partial t} + \bar{V}.\nabla(\rho w) = -\frac{\partial p}{\partial z}$$

For steady flow the partial time derivative vanishes, and we can write:

$$\bar{V}.\nabla(\rho u) = -\frac{\partial p}{\partial x}$$
$$\bar{V}.\nabla(\rho v) = -\frac{\partial p}{\partial y} \qquad (18)$$
$$\bar{V}.\nabla(\rho w) = -\frac{\partial p}{\partial z}$$

Expanding the gradient term, we get:

$$
\begin{aligned}
&\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \rho w \frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} \\
&\rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} + \rho w \frac{\partial v}{\partial z} = -\frac{\partial p}{\partial y} \\
&\rho u \frac{\partial w}{\partial x} + \rho v \frac{\partial w}{\partial y} + \rho w \frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z}
\end{aligned} \tag{19}
$$

Rearranging, we get:

$$
\begin{aligned}
&u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \\
&u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} \\
&u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}
\end{aligned} \tag{20}
$$

STREAMLINES DIFFERENTIAL EQUATIONS

The definition of a streamline in a flow is that it is parallel to the velocity vector. Hence the cross product of the directed element of the streamline and the velocity vector is zero:

$$
d\bar{s} \times \bar{V} = 0 \tag{21}
$$

where:

$$
\begin{aligned}
d\bar{s} &= dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z} \\
\bar{V} &= u\,\hat{x} + v\,\hat{y} + w\,\hat{z}
\end{aligned}
$$

The cross product can be expanded in the form of a determinant as:

$$
\begin{aligned}
d\bar{s} \times \bar{V} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ dx & dy & dz \\ u & v & w \end{vmatrix} \\
&= \hat{x}(w\,dy - v\,dz) + \hat{y}(u\,dz - w\,dx) + \hat{z}(v\,dx - u\,dy) \\
&= 0
\end{aligned} \tag{22}
$$

The vector being equal to zero, its components must be equal to zero yielding the differential equations for the streamline f(x, y, z) = 0, as:

$$
\begin{aligned}
w\,dy - v\,dz &= 0\\
u\,dz - w\,dx &= 0\\
v\,dx - u\,dy &= 0
\end{aligned}
\tag{23}
$$

EULER'S EQUATION

Multiplying the flow equations respectively by dx, dy and dz, we get:

$$
\begin{aligned}
u\frac{\partial u}{\partial x}dx + v\frac{\partial u}{\partial y}dx + w\frac{\partial u}{\partial z}dx &= -\frac{1}{\rho}\frac{\partial p}{\partial x}dx\\
u\frac{\partial v}{\partial x}dy + v\frac{\partial v}{\partial y}dy + w\frac{\partial v}{\partial z}dy &= -\frac{1}{\rho}\frac{\partial p}{\partial y}dy\\
u\frac{\partial w}{\partial x}dz + v\frac{\partial w}{\partial y}dz + w\frac{\partial w}{\partial z}dz &= -\frac{1}{\rho}\frac{\partial p}{\partial z}dz
\end{aligned}
\tag{24}
$$

Using the streamline differential equations, we can write:

$$
\begin{aligned}
u\frac{\partial u}{\partial x}dx + u\frac{\partial u}{\partial y}dy + w\frac{\partial u}{\partial z}dz &= -\frac{1}{\rho}\frac{\partial p}{\partial x}dx\\
u\frac{\partial v}{\partial x}dx + v\frac{\partial v}{\partial y}dy + w\frac{\partial v}{\partial z}dz &= -\frac{1}{\rho}\frac{\partial p}{\partial y}dy\\
u\frac{\partial w}{\partial x}dx + v\frac{\partial w}{\partial y}dy + w\frac{\partial w}{\partial z}dz &= -\frac{1}{\rho}\frac{\partial p}{\partial z}dz
\end{aligned}
\tag{25}
$$

The differentials of functions u = u(x, y, z), v = v(x, y, z), w = w (x, y, z) are:

$$
\begin{aligned}
du &= \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz\\
dv &= \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz\\
dw &= \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz
\end{aligned}
\tag{26}
$$

This allows us to write:

$$
\begin{aligned}
udu &= -\frac{1}{\rho}\frac{\partial p}{\partial x}dx \\
vdv &= -\frac{1}{\rho}\frac{\partial p}{\partial y}dy \\
wdw &= -\frac{1}{\rho}\frac{\partial p}{\partial z}dz
\end{aligned}
\qquad (27)
$$

Through integration we can write:

$$
\begin{aligned}
\frac{1}{2}d(u^2) &= -\frac{1}{\rho}\frac{\partial p}{\partial x}dx \\
\frac{1}{2}d(v^2) &= -\frac{1}{\rho}\frac{\partial p}{\partial y}dy \\
\frac{1}{2}d(w^2) &= -\frac{1}{\rho}\frac{\partial p}{\partial z}dz
\end{aligned}
\qquad (28)
$$

Adding the three last equations we get:

$$
\begin{aligned}
\frac{1}{2}d(u^2+v^2+w^2) &= -\frac{1}{\rho}\left(\frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial y}dy+\frac{\partial p}{\partial z}dz\right) \\
\frac{1}{2}d(V^2) &= -\frac{1}{\rho}dp
\end{aligned}
\qquad (29)
$$

From the last equation we can write a simple form of Euler's equation as:

$$
dp = -\rho V dV \qquad (30)
$$

Euler's equation applies to an inviscid flow with no body forces at steady-state. It relates the change in speed along a streamline dV to the change in pressure dp along the same streamline.

BERNOULLI EQUATION, INCOMPRESSIBLE FLOW

Considering the case of incompressible flow, we can use limit integration to yield:

$$
\begin{aligned}
&\int_{p_1}^{p_2} dp = -\rho\int_{V_1}^{V_2} V dV \\
&p_2 - p_1 = -\frac{\rho}{2}\left(V_2^2 - V_1^2\right) \\
&p_1 + \frac{1}{2}\rho V_1^2 = p_2 + \frac{1}{2}\rho V_2^2 = constant
\end{aligned}
\qquad (31)
$$

The relation between pressure p and speed V in an inviscid incompressible flow was enunciated in the form of Bernoulli's equation, first presented by Euler:

$$p + \frac{1}{2}\rho V^2 = constant \tag{32}$$

This equation is the most famous equation in fluid mechanics. Its significance is that when the velocity increases, the pressure decreases, and when the velocity decreases, the pressure increases.

The dimensions of the terms in the equation are kinetic energy per unit volume. Even though it was derived from the momentum conservation equation, it is also a relation for the mechanical energy in an incompressible flow. It states that the work done on a fluid by the pressure forces is equal to the change of kinetic energy of the flow. In fact it can be derived from the energy conservation equation of fluid flow.

The fact that Bernoulli's equation can be interpreted as Newton's second law or an energy equation illustrates that the energy equation is redundant for the analysis of inviscid, incompressible flow.

REFERENCES

1. John D. Anderson, Jr., "Fundamentals of Aerodynamics," 3rd edition, McGrawHill, 2001.

EXERCISES

1. From Euler's equation:

$dp = -\rho V dV$

Derive the expression for Bernoulli's law suggesting that the sum of the static and kinetic pressures is a constant between two points at steady-state in an inviscid flow without body forces.

2. A wind rotor airfoil is placed in the air flow at sea level conditions with a free stream speed of 10 m/s. The density at standard sea level conditions is 1.23 kg/m^3 and the pressure is 1.01 x 10^5 Newtons / m^2. At a point along the rotor airfoil the pressure is 0.90 x 10^5 Newtons / m^2. By applying Bernoulli's equation estimate the wind speed at this point.