ONE GROUP REACTOR THEORY

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1. INTRODUCTION

The steady state neutron diffusion theory is considered and is specialized to the situation of multiplying media. Specifically, the source term in the Helmholtz equation is expressed as a function of the fission medium's multiplication factor. This leads to an eigenvalue equation, characteristic, or criticality equation for multiplying media. With the appropriate boundary conditions, this derived equation is solved in spherical geometry for a critical reactor, and the results are compared to the experimental values of critical and subcritical assemblies.

2. STEADY STATE NEUTRON DIFFUSION EQUATION

The one group diffusion equation can be written as a neutron balance equation over an arbitrary volume V of a multiplying or non-multiplying medium as:

$$\frac{\partial \mathrm{n}}{\partial \mathrm{t}} = \left[\frac{\text{neutrons produced}}{\text{unit volume.unit time}}\right] - \left[\frac{\text{neutrons lost by absorption}}{\text{unit volume.unit time}}\right] - \left[\frac{\text{neutrons lost by leakage}}{\text{unit volume.unit time}}\right] \tag{1}$$

If the system is just critical then:

$$\text{Losses} = \text{Gains},$$

and:

$$\frac{\partial n}{\partial t} = 0 ,$$

for a steady state reactor operational condition.

The diffusion equation can be symbolically written as:

$$\frac{\partial \mathrm{n}}{\partial \mathrm{t}} = 0 = \mathrm{S} - \sum_{\mathrm{a}} \phi + \mathrm{D}\nabla^2\phi \tag{2}$$

where: ϕ is the neutron flux $[\frac{\mathrm{n}}{\mathrm{cm}^2.\mathrm{sec}}]$

$\sum_a$ is the macroscopic absorption coefficient [cm^{-1}],

D is the diffusion coefficient [cm].

3. DIFFUSION COEFFICIENT

From diffusion theory, the diffusion coefficient is expressed in terms of the macroscopic scattering cross section as:

$$D = \frac{1}{3\sum_s} \tag{3}$$

where: $\sum_s$ is the macroscopic scattering cross section.

However, from the more advanced transport theory it can be expressed in terms of the macroscopic transport, scattering and absorption cross sections as:

$$\begin{aligned} D &= \frac{1}{3[\sum_a + \sum_s(1-\bar{\mu}_0)]} \\ &= \frac{1}{3[\sum_a + \sum_{tr}]} \end{aligned} \tag{3'}$$

where: $\sum_{tr}$ is the macroscopic transport cross-section,

$$\Sigma_{tr} = \Sigma_s(1-\bar{\mu}_0)$$
$$\lambda_{tr} = \frac{1}{\Sigma_{tr}} = \frac{1}{\Sigma_s(1-\bar{\mu}_0)}$$

λ_{tr} is the transport mean free path

$\sum_a$ is the macroscopic absorption cross-section,

$\sum_s$ is the macroscopic scattering cross-section,

$\bar{\mu}_0$ is the average cosine of the scattering angle in the laboratory system.

In a weakly absorbing medium where $\sum_a << \sum_s$, D becomes:

$$D = \frac{1}{3\Sigma_{tr}} = \frac{\lambda_{tr}}{3} \tag{3'}$$

where λ_{tr} is the transport mean free path [cm].

In a reactor containing a number of different materials, an average value of $(1-\bar{\mu}_0)$ is calculated using the weighted sum:

$$(1-\bar{\mu}_0)_{\text{reactor}} = \sum_{i=1}^{n} \frac{(1-\bar{\mu}_0)_i \, \Sigma_{si}}{\Sigma_{s,reactor}} \tag{4}$$

where: n is the total number of elements present in the reactor.

The average cosine of the scattering angle is given by:

$$\bar{\mu}_0 = \overline{\cos\theta} = \frac{2}{3A} \tag{5}$$

where: θ is the neutron scattering angle,

A is the mass number of the scattering isotope, [amu].

4. THE INFINITE MEDIUM MULTIPLICATION FACTOR AND THE SOURCE TERM

In an infinite medium, the flux assumes a constant value, no gradient exists, and hence there is no neutron leakage and we can define an infinite medium multiplication factor as:

$$k_\infty = \frac{\text{neutrons produced in current fission generation}}{\text{neutrons absorbed in previous fission generation}}$$

The infinite medium multiplication factor can be expressed in terms of the four-factor formula:

$$\mathrm{k}_\infty = \eta\varepsilon\mathrm{pf} \tag{6}$$

where: η is the regeneration factor,

ε is the fast fission factor,

p is the resonance escape probability,

f is the fuel utilization factor.

The fuel utilization factor, for a constant flux value in a homogeneous medium, is defined as:

$$
\begin{aligned}
f &= \frac{\text{rate of neutrons absorptions in the fuel}}{\text{total rate of neutron absorptions in fuel, moderator, structure, etc.}} \\
&= \frac{\sum_{aF} \phi}{\sum_{aF} \phi + \sum_{aM} \phi + \sum_{aS} \phi + \ldots} \\
&= \frac{\sum_{aF}}{\sum_{aF} + \sum_{aM} + \sum_{aS} + \ldots}
\end{aligned} \tag{7}
$$

where: M stands for the moderator,
F stands for the fuel,
S stands for the structure.

The resonance escape probability p is that fraction of the fast neutrons, which slow down to thermal energies past the cross section energy resonance region without being absorbed.

The fast fission factor ε is the ratio of the total fissions produced from fast and thermal neutrons to the fissions produced by thermal neutrons.

The regeneration factor η is:

$$
\eta = \frac{\text{neutrons produced from fission}}{\text{neutrons absorbed in fuel}}
$$

For U^{235},

$$
\begin{aligned}
\eta &= \nu \frac{\sum_f \phi}{\sum_{aF} \phi} \\
&= \nu \frac{\sum_f}{\sum_f + \sum_\gamma} \\
&= \nu \frac{N\sigma_f}{N\sigma_f + N\sigma_\gamma} \\
&= \nu \frac{\sigma_f}{\sigma_f + \sigma_\gamma}
\end{aligned} \tag{8}
$$

where: ν is the average number of neutrons emitted per fission,
$\sum_f, \sum_\gamma$ are the macroscopic fission and radiative capture cross sections respectively.

If we designate the ratio of the radiative capture to fission microscopic cross sections as the capture to fission ratio:

$$
\alpha = \frac{\sigma_\gamma}{\sigma_f} \tag{9}
$$

then:

$$\eta = \frac{\nu}{1+\alpha} \tag{10}$$

Since Eqn. 2 applies for neutrons of a single energy that are thermal, we can write for the thermal neutrons source term S:

$$S = k_\infty \ell_f \sum_a \phi \tag{11}$$

where: ℓ_f is the fast neutron non-leakage probability.

For the transient state, Eqn. 2 can be written as:

$$\frac{\partial n}{\partial t} = \frac{1}{\mathrm{v}}\frac{\partial(\mathrm{nv})}{\partial \mathrm{t}} = \frac{1}{\mathrm{v}}\frac{\partial \phi}{\partial \mathrm{t}} = D\nabla^2\phi - \sum_a \phi + k_\infty \ell_f \sum_a \phi$$

Dividing into $\sum_a$:

$$\frac{1}{\mathrm{v}\sum_a}\frac{\partial \phi}{\partial \mathrm{t}} = L^2\nabla^2\phi - \phi + k_\infty \ell_f \phi \tag{12}$$

where: $\mathrm{L}^2 = \frac{\mathrm{D}}{\sum_a}$ is the thermal diffusion area [cm^2],

L is the diffusion length [cm].

Rearranging, we get after dividing by $L^2\phi$:

$$\frac{1}{\mathrm{vL}^2\sum_a}\frac{1}{\phi}\frac{\partial \phi}{\partial \mathrm{t}} - \frac{\nabla^2\phi}{\phi} = \frac{\mathrm{k}_\infty \ell_\mathrm{f} - 1}{\mathrm{L}^2}$$

$$\frac{1}{\mathrm{vL}^2\sum_a}\frac{1}{\phi}\frac{\partial \phi}{\partial \mathrm{t}} + \mathrm{B_g}^2 = \mathrm{B_m}^2$$

where: $\mathrm{B_m}^2 = \frac{\mathrm{k}_\infty \ell_\mathrm{f} - 1}{\mathrm{L}^2}$ is denoted as the "material buckling," since it a function of the materials composition,

$\mathrm{B_g}^2 = -\frac{\nabla^2\phi}{\phi}$ is known as the "geometrical buckling," since it is a function of the geometry.

The terminology for the geometrical buckling arises in the field of structural mechanics where it describes the degree of deformational buckling of a column under external forces applied to its ends.

5. THE CRITICALITY EQUATION

For steady state, $\frac{\partial \phi}{\partial t} = 0$, and Eqn. 12 implies that for criticality and a self-sustained chain reaction, the following condition where the geometric buckling is equal to the material buckling must apply:

$$-\frac{\nabla^2 \phi}{\phi} = \frac{k_\infty \ell_f - 1}{L^2}$$

$$B_g^{\,2} = B_m^{\,2}$$

In general:

$$B^2 = B_g^{\,2} = B_m^{\,2},$$

where: B^2 is called the buckling in short.

As a criticality condition, the geometrical and material bucklings should thus be equal for a nuclear reactor to become critical and be able to establish a steady-state self-sustained neutron chain reaction.

This criticality condition expresses the need to match the material properties of the medium expressed in the equation for the material buckling, to the geometry of the system expressed by its geometrical buckling.

Rearranging the expression for the material buckling, we get:

$$1 = \frac{k_\infty \ell_f}{(1 + L^2 B^2)}$$

The last equation represents a just critical system. In general though, we can define an effective multiplication factor as:

$$k_{eff} = k_\infty \ell_f \frac{1}{1 + L^2 B^2} = k_\infty \ell_f \ell_{th} \qquad (13)$$

where: ℓ_{th} is the thermal neutrons non-leakage probability,

$$\ell_{th} = \frac{1}{1 + L^2 B^2}$$

where: ℓ_f is the fast neutrons non-leakage probability

$k_{eff} < 1$, for a subcritical reactor,

$k_{eff} > 1$, for a supercritical reactor,

$k_{eff} = 1$, for a just critical reactor.

6. SOLUTION OF THE UNREFLECTED REACTOR EIGENVALUE EQUATION

From the definition of the geometrical buckling:

$$B_g^2 = -\frac{\nabla^2 \phi}{\phi},$$

we get:

$$\nabla^2 \phi = -B_g^2 \phi$$

In spherical geometry, expressing the Laplacian Operator, we get

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\phi}{dr}\right) = \frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} = -B_g^2 \phi$$

The solution of this equation is:

$$\phi(r) = \frac{A\cos(B_g r)}{r} + \frac{C\sin(B_g r)}{r}$$

If the flux is finite throughout the reactor, then A = 0 (since: $\frac{\cos 0}{0} = \frac{1}{0} = \infty$) leading to the unphysical situation of an infinite value for the flux, and we are compelled to choose as a physical solution:

$$\phi(r) = \frac{C\sin(B_g r)}{r}$$

As $r \to 0$, from l'Hospital's Rule the value of the flux at the center of the reactor is:

$$\phi_0 = \text{limit}_{r \to 0} \frac{CB_g \cos(B_g r)}{1} = CB_g$$

and:

$$C = \frac{\phi_0}{B_g},$$

from which:

$$\phi(r) = \phi_0 \frac{\sin(B_g r)}{(B_g r)}$$

where: ϕ_0 is the flux at the reactor center.

A second boundary condition is that of the flux vanishing at the extrapolated radius:

$$\mathrm{R_{ex}} = \mathrm{R} + \mathrm{d}$$

where: d is the extrapolated length:

$$d = \frac{2}{3}\lambda_{tr}$$

From transport theory, a better estimate of d is usually used:

$$d = 0.71\,\lambda_{tr}.$$

Applying this condition, we get:

$$\mathrm{B_g R_{ex}} = \mathrm{n}\,\pi,\ \ \mathrm{n} = 1, 2, \ldots$$

For the main harmonic solution or n = 1 eigenvalue,

$$\mathrm{B_g} = \frac{\pi}{\mathrm{R_{ex}}},$$

and the flux distribution becomes:

$$\phi(\mathrm{r}) = \phi_0 \frac{\sin\left(\frac{\pi \mathrm{r}}{\mathrm{R_{ex}}}\right)}{\left(\frac{\pi \mathrm{r}}{\mathrm{R_{ex}}}\right)} \qquad (14)$$

7. EFFECT OF REACTOR POWER, SEMI INFINITE SLAB REACTOR

The effect of the reactor power level can be included in the magnitude constant of integration to the reactor criticality equation. Mathematically, for the eigen-value equation:

$$\nabla^2\phi = -B_g^2\phi$$

the solution for the flux multiplied by any constant is still a solution to the equation. Physically, this means that we can operate the nuclear reactor at any power level we wish, provided enough cooling to extract the heat generated is provided. Otherwise, if the energy release is not extracted, this would lead to melting and possibly evaporation of the reactor core.

For a hypothetical semi-infinite slab reactor of thickness a in the x direction with the origin taken at the center of the slab, and infinite in extent in the y and z directions, with a power produced per unit area P, the reactor equation can be written in one dimensional Cartesian coordinates as:

$$\frac{d^2\phi(x)}{dx^2} = -B^2\phi(x)$$

Its solution is harmonic as:

$$\phi(x) = A\cos(Bx) + C\sin(Bx)$$

Since the flux cannot be zero and reaches a maximum at the origin we reject the sin (Bx) solution implying that C = 0:

$$\phi(x) = A\cos(Bx)$$

Applying the boundary condition of a vanishing flux at the boundaries + a/2 and – a/2, and neglecting the extrapolated length for a large reactor,

$$B\frac{a}{2} = n\frac{\pi}{2},\ n = 1,3,5,\ ...$$

$$Ba = n\pi$$

$$B = n\frac{\pi}{a}$$

Considering the fundamental mode or main harmonic n = 1, we can write:

$$\phi(x) = A\cos(\frac{\pi}{a}x)$$

The reactor power per unit area can be expressed as:

$$\begin{aligned} P &= \int_V \Sigma_f \, \phi E_f dV \\ &= \Sigma_f \, E_f A \int_{x=-\frac{a}{2}}^{x=+\frac{a}{2}} \cos(\frac{\pi}{a}x).1.dx \\ &= \Sigma_f \, E_f A \frac{a}{\pi}\left[\sin(\frac{\pi}{a}x)\right]_{-\frac{a}{2}}^{+\frac{a}{2}} \\ &= \Sigma_f \, E_f A \frac{a}{\pi}[\sin(\frac{\pi}{2}) - \sin(-\frac{\pi}{2})] \\ &= \Sigma_f \, E_f A \frac{a}{\pi}.2 \end{aligned}$$

where: E_f is the extractable energy release per fission event = 190 [MeV/fission]

The constant of integration A can be expressed as:

$$A = \frac{\pi P}{2aE_f \Sigma_f}$$

and the flux distribution will be dependent on the power per unit area P as:

$$\phi(x) = \frac{\pi P}{2aE_f \Sigma_f}\cos\left(\frac{\pi}{a}x\right),$$

suggesting that the reactor can be operated at any desired power level P, provided enough cooling is provided; a limiting property of the used materials at high temperatures that it shares with any other power producing engine, including a fossil power plant boiler or an automobile engine.

8. THE MODIFIED ONE GROUP CRITICALITY EQUATION

Defining a fast neutrons diffusion area also referred to as neutron age τ [cm^2], then in analogy to the expression for the thermal non-leakage probability:

$$\ell_{th} = \frac{1}{1 + L^2 B^2},$$

we can write an expression for the fast non-leakage probability as:

$$\ell_f = \frac{1}{1 + \tau B^2}.$$

Thus Eqn.13 for the effective multiplication factor can be written for a critical system as:

$$k_{eff} = k_\infty \frac{1}{(1 + L^2 B^2)} \cdot \frac{1}{(1 + \tau B^2)} = k_\infty \ell_{th} \ell_f$$

$$k_{eff} = k_\infty \frac{1}{1 + (L^2 + \tau) B^2 + L^2 \tau B^4}$$

Now, $B_g^2 = (\frac{\pi}{R_{ex}})^2$, so that we can neglect B^4 with respect to B^2 in the case of a large reactor with a large radius R, and write the modified one group theory criticality equation:

$$k_{eff} = \frac{k_\infty}{1 + M^2 B^2} \qquad (15)$$

where: $M^2 = L^2 + \tau$ is designated as the migration area.

$B^2 = B_g^{\ 2} = B_m^{\ 2}$ is the buckling.

9. DETERMINATION OF REACTOR'S CRITICAL DIMENSION

If we now write the expression for the buckling, and rearrange to get:

$$B_m^2 = \frac{k_\infty - 1}{M^2} = (\frac{\pi}{R_{ex}})^2$$

From which:

$$R_{ex} = \pi\sqrt{\frac{M^2}{k_\infty - 1}}$$

and:

$$R_c = R_{ex} - d = \frac{\pi M}{\sqrt{k_\infty - 1}} - d \qquad (16)$$

is obtained as the reactor critical radius.

10. THE CRITICAL RADIUS OF AN UNREFLECTED AND UNMODERATED CRITICAL FISSILE SPHERE

Table 1. One Group Fast Group Constants, from ANL-5800 [4].

Nuclide	Neutrons per fission event ν	Fission cross section σ_f [barn]	Capture cross section σ_c [barn]	Transport cross section σ_{tr} [barn]
Plutonium^{239}	2.98	1.85	0.260	6.8
Uranium^{235}	2.6	1.40	0.250	6.8
Uranium^{238}	2.6	0.095	0.16	6.9
Fe	-	-	0.006	2.7
Na	-	-	0.0008	3.3
Al	-	-	0.002	3.1

If we consider an unmoderated fissile reactor, the neutron spectrum would be a fast one, since no moderator is present to moderate the energy of the neutrons to thermal energy. The cross section data must be considered as weighed by a fast neutron spectrum instead of a thermal neutron spectrum in the case of moderated systems. Such fast one group constants differ from one source in the literature to another because of the flux weighting procedure that is adopted. One such set is shown in Table 1.

11. CRITICALITY OF BARE, UNREFLECTED AND UNMODERATED FAST REACTOR

As an example, we shall calculate the critical radius, volume, and mass of a sphere of U^{235} , given the following fast spectrum data from Wirtz [3]:

$$\rho(^{235}U) = 18.75\ [gm/cm^3]$$
$$\nu\sigma_f = 5.297 \text{ neutron.b}$$
$$\sigma_a = 2.844\ b$$
$$\sigma_{tr} = 8.246\ b$$

These data are chosen for a fast neutron spectrum since no moderator in included to slow the neutrons down. We shall use one-group theory, and neglect the extrapolation distance. The solution then proceeds as follows.

Equating the material and geometrical bucklings for a sphere in one-group theory:

$$B_g^2 = (\frac{\pi}{R_c})^2 = \frac{k_\infty - 1}{L_f^2} = B_m^2$$

Note that for the modified two-group theory L^2_f is to be replaced by the migration area M^2. From the last equation the critical radius is:

$$R_c = \pi\sqrt{\frac{L_f^2}{k_\infty - 1}}$$

where: infinite medium multiplication factor $k_\infty = \eta\varepsilon pf$

fast neutrons diffusion area $L_f^2 = \frac{D}{\sum_a}$

fast neutrons diffusion coefficient $D = \frac{1}{3\sum_{tr}}$

For a pure U^{235} sphere the fuel utilization factor is:

$$f = \frac{\sum_{aF}}{\sum_a} \approx \frac{\sum_a}{\sum_a} = 1.$$

Also $p \approx \varepsilon \approx 1$ if we neglect fast fissions and resonance absorptions.

Thus:

$$k_\infty \approx \eta \approx \frac{\nu\sigma_f}{\sigma_a} = \frac{5.297}{2.844} = 1.86$$

$$\sum_{tr} = N\sigma_{tr} = \sigma_{tr}\frac{\rho.Av}{M} = \frac{18.75 \times 0.6021\text{x}10^{+24}}{235} 8.246\text{x}10^{-24} = 0.396[cm^{-1}]$$

$$\sum_a = N\sigma_a = \sigma_a \frac{\rho.Av}{M} = \frac{18.75 \times 0.6021\text{x}10^{+24}}{235} \times 2.844\text{x}10^{-24} = 0.137[cm^{-1}]$$

$$L_f^2 = \frac{D}{\sum_a} = \frac{1}{3\sum_{tr}\sum_a} = \frac{1}{3 \times 0.396 \times 0.137} = 6.144[cm^2]$$

Thus the critical radius is:

$$R_c = \pi\sqrt{\frac{6.144}{1.86-1}} = \pi\sqrt{7.144} = 8.397[cm]$$

The corresponding critical volume is:

$$V_c = \frac{4\pi R_c^3}{3} = 2,480.054\ [cm^3]$$

And the critical mass becomes:

$$M_c = V_c\rho = 2,480.054 \times 18.75 = 46,501.013\ gm = 46.5\ kg$$

Table 2. Effective multiplication factor of unreflected and reflected fast reactor assemblies.

Assembly Core	Reflector	Core Radius [cm]	k_{eff}
U^{233}	-	5.965	1.0115
Godiva, U^{235}	-	8.710	0.9912
Jezebel, Pu^{239}	-	6.285	1.0039
37.5 % U^{235}	-	14.57	0.9855
16.7 % U^{235}	7.6 cm U	20.32	0.9893
U^{235}	1.8 cm U	7.725	0.9907
U^{235}	8.9 cm U	6.391	0.9939
Topsy, U^{235}	U	6.045	0.9907
U^{235}	5.1 cm Fe	7.3900	0.9756
U^{235}	4.6 cm Th	7.800	0.9905
ZPR-III 48	30 cm U	47.4200	1.0160

This compares to the radius of 8.71 cm and the critical mass of 48.8 kg for the Godiva critical experiment, which is a 93.9 percent U^{235} enriched system Oralloy compostion. The core radii of different fast critical assemblies are shown in Table 2. The Zero Power Reactor ZPR-III 48 experiment simulated a fast reactor with a core consisting of carbides of uranium^{235} and

plutonium^{239}. Sodium was used as a coolant, iron as the structural material, and uranium was used as a reflector.

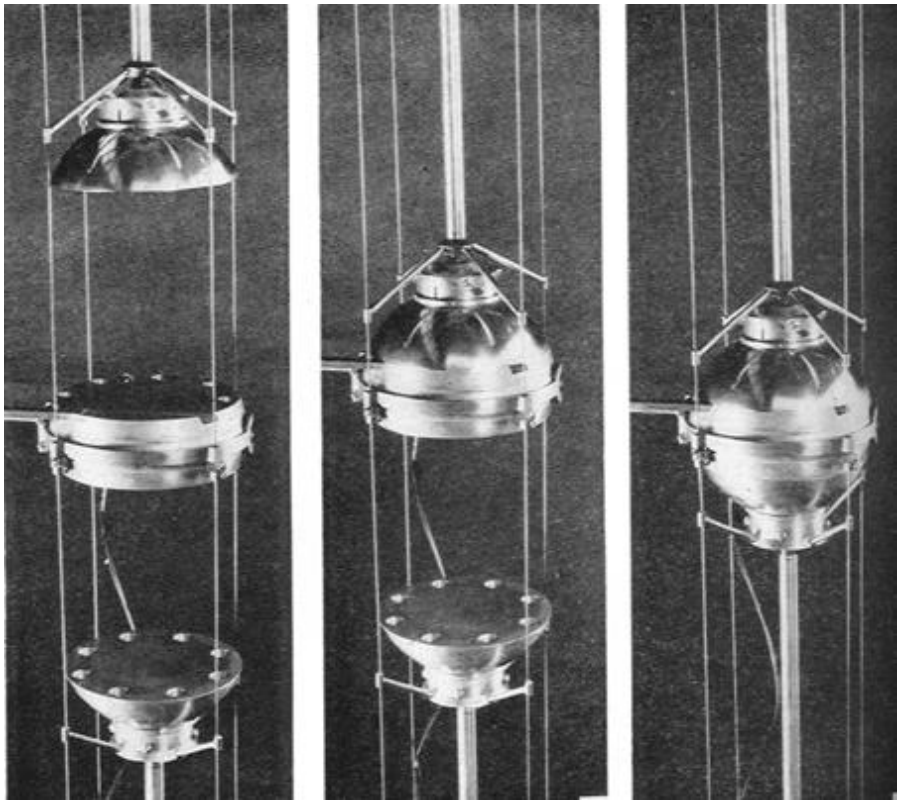

Figure 1. The Jezebel critical plutonium sphere experiment.

12. THE JEZEBEL CRITICAL PLUTONIUM SPHERE

Critical experiments are conducted to experimentally check the calculated values of the critical masses. Figure 1 shows a critical experiment for a plutonium^{239} sphere, designated as Jezebel, where the sphere is divided into three parts remotely brought together along sliding wires.

The Jezebel experiment, set up in 1954-1955, was a spherical bare unreflected homogeneous assembly of Pu that is stabilized with 3 – 4.5 atomic percent rare earth element gallium into the delta phase at room temperature. In 1956 its measured critical mass was determined as 16.45 +/- 0.04 kg. This value was updated in 1969 to 16.57 +/- 0.10 kg. A later estimation by the X-Division at the Los Alamos National Laboratory (LANL) in 2016 establishes a value of 16.624 +/- 0.065 kg.

Solid plutonium is most malleable in the delta phase. This phase is the most suitable for its fabrication. The delta phase of pure plutonium is stable only between 600 and 700 kelvin well above the 293 kelvin of room temperature. The delta phase can be stabilized at room temperature with gallium, offering easier machinability, higher corrosion resistance and a melting temperature

at 650 ^{o}C. Unalloyed plutonium melts at a relatively low temperature around 640 ^{o}C to yield a liquid of higher density than the solid form from which it melts. The elastic properties of the delta face centered cubic (fcc) phase of plutonium are highly directional or anisotropic. This means that the elasticity of the metal varies widely along different crystallographic directions by a factor of six to seven.

The delta fcc form of plutonium is stable at high temperatures and the highly symmetric fcc structure can be retained at room temperature by adding 2 atomic percent of an alloy rare earth metal such as gallium.

13. EFFECT OF EXTRAPOLATION DISTANCE

If the extrapolation distance is not neglected in the treated example, then:

$$d \approx 0.71\lambda_{tr} = \frac{0.71}{\Sigma_{tr}} = \frac{0.71}{0.396} = 1.793\ cm$$

and since in this case the extrapolated radius is:

$$R_{ex} = R_c + d = 8.397\,cm$$

Thus the critical assembly radius is:

$$R_c = 8.397 - 1.793 = 6.604\,cm$$

The critical volume will be:

$$V_c = \frac{4\pi(6.604)^3}{3} = 1,206.451\ \ cm^3$$

and the critical mass can be calculated as:

$$M_c = 1,206.451 \times 18.75 = 22,620.962\,gm = 22.6\,kgs$$

This is much different from the previously determined value, almost by a factor of one half, emphasizing the need to account for the extrapolation distance in small reactors. In large reactors it can be neglected. In addition, more sophisticated methods than one-group diffusion theory, such as transport theory are needed to theoretically determine the accurate sizes of critical assemblies.

14. CRITICALITY OF MODERATED HOMOGENEOUS REACTORS WITH DIFFERENT GEOMETRIES

Let us consider a large reactor that is composed of a homogeneous mixture of pure U^{235} and graphite as carbon. There is one atom of U^{235} for every 10,000 atoms of carbon, or:

$$\frac{N_C}{N_U} = 10^4 .$$

The migration area is $M^2 = 3{,}040\ cm^2$. We ask ourselves the following questions:

1. What is the material buckling ?
2. If the reactor is a bare cylinder with a height equal twice the radius, what is the critical radius ?
3. If the reactor is a bare sphere, what is its critical radius ?
4. If the reactor is a cube, what is its critical radius ?

We shall compare the required critical volumes for the three last cases and estimate the U^{235} mass required for criticality in each case. We proceed with the solution in the following way.

From the modified one-group theory, the material buckling is given by:

$$B_m^2 = \frac{k_\infty - 1}{M^2}, \quad k_\infty = \quad \eta\varepsilon pf$$

where: M^2 is the migration area.

Neglecting fast fissions, then $\varepsilon \approx 1$, and neglecting resonance absorption, then $p \approx 1$.

The regeneration factor can be written as:

$$\eta = \nu \frac{\sigma_f}{\sigma_a} = \frac{\nu}{1+\alpha} = 2.08$$

The thermal utilization factor is:

$$f = \frac{\sum_{aF}}{\sum_{aF} + \sum_{aC}} = \frac{1}{1 + \frac{N_C}{N_F}\frac{\sigma_{aC}}{\sigma_{aF}}}$$

Substituting:

$$\sigma_{ac} = 3.4 \times 10^{-6}\ b$$
$$\sigma_{aF} = 2.844\ b$$
$$\frac{N_C}{N_U} = \frac{N_C}{N_F} = 10^4$$

we get:

$$f = 0.988.$$

From which:

$$k_\infty \approx 2.08 \times 1 \times 1 \times 0.988 = 2.06 .$$

The material buckling becomes:

$$B_m^2 = \frac{2.06 - 1}{3040} = 3.49 \times 10^{-4} \text{cm}^{-2}$$

For an unreflected bare cylindrical core, neglecting the extrapolation lengths, the geometrical buckling is:

$$B_g^2 = (\frac{\pi}{H})^2 + (\frac{2.405}{R})^2$$

Since the height of the reactor H = 2R, we get:

$$B_g^2 = (\frac{\pi}{2R})^2 + (\frac{2.405}{R})^2 = \frac{8.25}{R^2}$$

Equating the geometrical buckling to the material buckling the critical radius is:

$$R_c = (\frac{8.25}{3.49 \times 10^{-4}})^{1/2} = 153.75 \text{ cm}$$

For an unreflected bare sphere:

$$B_g^2 = (\frac{\pi}{R})^2,$$

and:

$$R_c = (\frac{\pi^2}{3.49 \times 10^{-4}})^{1/2} = 168.17 \text{cm}$$

For a cube:

$$B_g^2 = 3 \times (\frac{\pi}{a})^2,$$

and:

$$a_c = \left(\frac{3\pi^2}{3.49\times10^{-4}}\right)^{1/2} = 291.8\text{cm}$$

The critical volumes are:

$$\begin{aligned} V_c(\text{cylinder}) &= \pi R^2 H = \pi R^2.2R_c = 2\pi R_c^3 &&= 2.28\times10^7\text{cm}^3 \\ V_c(\text{sphere}) &= \frac{4\pi R_c^3}{3} &&= 1.99\times10^7\text{cm}^3 \\ V_c(\text{cube}) &= a_c^3 &&= 2.47\times10^7\text{cm}^3 \end{aligned}$$

We can notice that even though:

$$R_c(\text{cylinder}) < R_c(\text{sphere}) < a_c(\text{cube}),$$

we have:

$$V_c(\text{sphere.}) < V_c(\text{cylinder}) < V_c(\text{cube}).$$

The mass of uranium needed for criticality in each case can be calculated as follows:

$$\frac{N_C}{N_U} = 10^4 = \frac{\frac{\rho_C (V - V_U) A_V}{12}}{\frac{\rho_U V_U A_V}{235}}$$

From which:

$$10^4 \cdot \frac{\rho_U}{\rho_C} \cdot \frac{12}{235} \cdot V_U = V - V_U$$

and:

$$g_U = \rho_U V_U = \frac{\rho_U V}{1 + 10^4 \cdot \frac{\rho_U}{\rho_C} \cdot \frac{12}{235}}$$

where: g_U is the weight of ^{235}U in volume V [gms],
V_U is the volume occupied by ^{235}U,

$$\rho_C = 1.60\ \frac{gm}{cm^3},$$

$$\rho_U = 19.1\ \frac{gm}{cm^3}.$$

Substituting for the constants:

$$g_U = 3.13\times10^{-3}V.$$

From which:

$$(g_U)_{\text{cylinder}} = 7.14\times10^{4}\,\text{gm}$$
$$(g_U)_{\text{sphere}} = 6.23\times10^{4}\,\text{gm}$$
$$(g_U)_{\text{cube}} = 7.73\times10^{4}\,\text{gm}$$

We can notice that:

$$(g_U)_{sphere} < (g_U)_{cylinder} < (g_U)_{cube}$$

This implies that a sphere is the optimal geometry from the point view of minimum fissile mass requirements for attaining criticality.

15. CRITICALITY OF REACTOR CORE WITH AN INFINITE REFLECTOR

Let us consider a spherical reactor of radius R and surrounded by an infinite reflector. We shall use the modified one-group theory to find the corresponding criticality conditions and the flux distribution in the core and in the reflector. A reflector is considered "infinite" if its thickness is equal to a large number of neutron mean free paths in it.

The modified one-group thermal diffusion equations in the core and reflector are:

$$D_C\nabla^2\phi_C - \Sigma_{ac}\phi_C + k_\infty\Sigma_{ac}\phi_C = 0$$
$$D_r\nabla^2\phi_r - \Sigma_{ar}\phi_r = 0$$

The solution in the core is:

$$\phi_C = \frac{C\sin Br}{r} = \phi_0\frac{\sin Br}{Br},$$

where:

$$B^2 = \frac{k_\infty - 1}{M^2}$$

The solution in the reflector is:

$$\phi_r = \frac{Ae^{-r/L_r}}{r} + \frac{Fe^{+r/L_r}}{r}$$

where:

$$L_r = \frac{D_r}{\Sigma_{ar}}.$$

For an infinite reflector and a finite flux, F = 0.

Applying the continuity of the flux and the current using Fick's law at the interface between the core and the reflector r = R, we get:

$$\phi_0 \frac{\sin BR}{BR} = \frac{Ae^{-r/L_r}}{R}$$

$$-D_c \frac{\phi_0}{B}\left(\frac{B\ \cos\ BR}{R} - \frac{\sin\ BR}{R^2}\right) = D_r Ae^{-R/L_r}\left(\frac{1}{L_r R} + \frac{1}{R^2}\right)$$

Dividing both equations we get:

$$-D_C\left(B\ \cot BR - \frac{1}{R}\right) = D_r\left(\frac{1}{L_r} + \frac{1}{R}\right)$$

$$BR\ \cot BR = 1 - \frac{D_r}{D_c}\left(\frac{R}{L_r} + 1\right)$$

This is a transcendental equation that can be solved graphically or by an iterative numerical method to obtain the critical radius, volume and mass, as well as the flux distribution in the core and reflector.

16. CRITICALITY OF REACTOR CORE WITH A FINITE SIZE REFLECTOR

A reflector of a scattering material surrounding a fissile core will reflect the neutrons that are leaking from the surface. This leads to a smaller critical mass for the fissile core. For a finite

reflector of thickness T, the magnitude of $C \neq 0$, and it is more convenient to write the solution in the reflector as:

$$\phi_r = \frac{A\cosh(r/L_r)}{r} + \frac{C\sinh(r/L_r)}{r}.$$

Applying the condition for the vanishing of the flux at r = R+T+d, we get:

$$C = -A\frac{\cosh\frac{R+T+d}{L_r}}{\sinh\frac{R+T+d}{L_r}} = -A\coth\frac{R+T+d}{L_r},$$

thus:

$$\begin{aligned}\phi_r &= \frac{A\cosh(r/L_r)}{r} - A\cosh\frac{R+T+d}{L_r}\cdot\frac{\sinh(r/L_r)}{r} \\ &= \frac{\mathrm{A}}{\mathrm{r}\sinh\frac{R+T+d}{L_r}}\left(\sinh\frac{R+T+d}{L_r}\cdot\cosh\frac{\mathrm{r}}{\mathrm{L_r}} - \cosh\frac{R+T+d}{L_r}\cdot\sinh\frac{\mathrm{r}}{\mathrm{L_r}}\right) \\ &= \frac{\mathrm{A}}{\mathrm{r}\sinh\frac{R+T+d}{L_r}}\sinh\frac{R+T+d-r}{L_r}\end{aligned}$$

Applying the conditions of continuity of the flux and current at r = R, we get:

$$\phi_0\frac{\sin BR}{BR} = \frac{A}{R\sinh\frac{R+T+d}{\mathrm{L_r}}}\sinh\frac{T+d}{L_r}$$

$$-D_c\frac{\phi_0}{B}\left(\frac{B\cos BR}{R} - \frac{\sin BR}{R^2}\right) = \frac{AD_r}{\sinh\frac{R+T+d}{\mathrm{L_r}}}\left(\frac{\cos\frac{(T+d)}{L_r}}{L_r R} + \frac{\sinh\frac{(T+d)}{L_r}}{R^2}\right)$$

Dividing both equations:

$$-\mathrm{D_c}(\mathrm{B}\cot\mathrm{BR} - 1/\mathrm{R}) = \mathrm{D_r}\left(\frac{\coth\frac{\mathrm{T+d}}{\mathrm{L_r}}}{\mathrm{L_r}} + \frac{1}{\mathrm{R}}\right)$$

Thus we get as a criticality condition for a reflector of thickness T, a transcendental equation to be solved graphically or numerically for the critical dimensions and flux distributions:

$$\cot BR = \frac{1}{BR}(1-\frac{D_r}{D_c}) - \frac{D_r}{D_c}\cdot\frac{1}{BL_r}\cdot\coth\frac{T+d}{L_r} \ .$$

17. STANDING-WAVE FAIL-SAFE REACTOR CORE

Systems engineers have a maxim that: “If a system is not designed to be fail-safe, tested under all combinations of extreme conditions, and operated perfectly, it will fail.” Hence it is mandatory that nuclear reactors designs must follow this maxim both at the design and the operational stages. Murphy’s Law: “If anything can go wrong, it will,” or: “Anything that can go wrong, will go wrong,” forces us to infer that our engineering systems will eventually fail if they were not developed so as to not fail in the first place. We thus attempt the consideration of a definitely fail-safe reactor design.

Consider a spherical, or a cylindrical reactor core with core radius R surrounded with an infinite reflector. If the core infinite medium multiplication factor is chosen to be exactly unity:

$$k_\infty = 1,$$

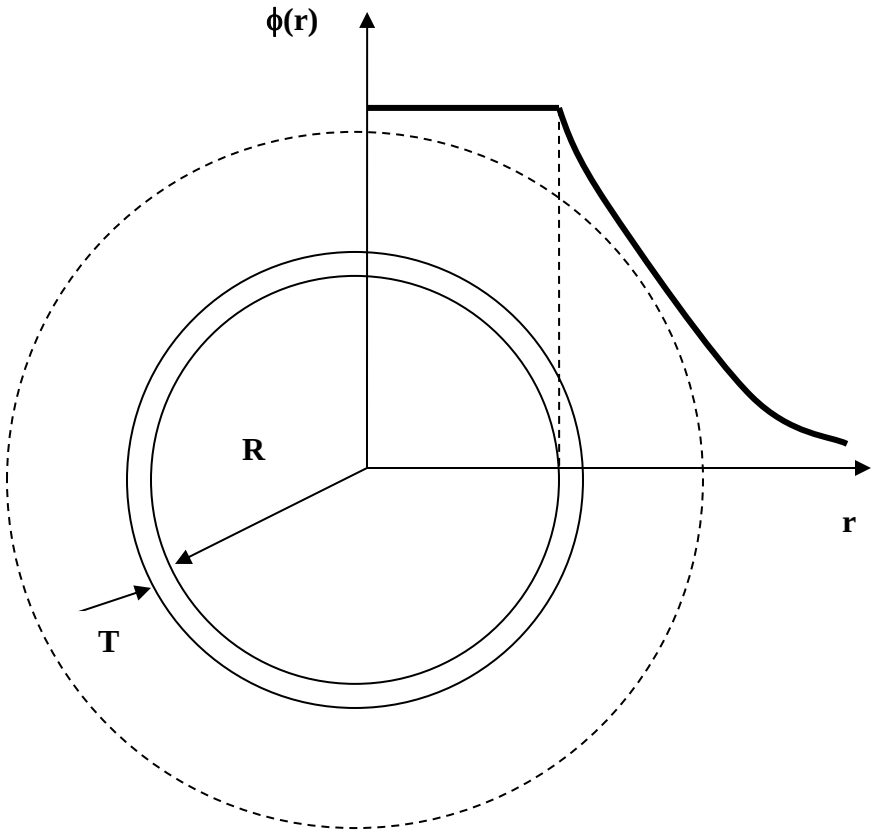

Figure 2. Geometry of a spherical or cylindrical core with an infinite reflector and a neutron source reflector interface.

the system would be essentially subcritical even with the presence of the reflector because of the leakage from the core to the reflector leading to a value of the effective multiplication factor of less than unity:

$$k_{eff} < 1$$

This would be a desirable inherently fail-safe situation encountered, for instance, during the rocket launch of a reactor into space or during its transport. Once the launch is safely completed the system can be made critical or armed if a neutron source of fissile material such as U^{235} is introduced to displace a void or an absorbing layer (for added safety) between the core and the reflector.

This can also be the basis of a fail-safe fission reactor configuration with a fission region neutron source or a neutron source from a DD or DT fusion reactor or an accelerator-driven

spallation system. The system falls automatically into a subcritical configuration once the neutron source is absent.

Assuming a thin thickness T of fissile material with a macroscopic absorption cross section Σ_a, and macroscopic fission cross section Σ_f, the neutron source introduces a net current at the interface equal to:

$$\begin{aligned} J_{\substack{neutron \\ source}} &= \nu \sum_f \phi_c(R).T - \sum_{aF} \phi_c(R).T \\ &= \nu \frac{\sum_f}{\sum_{aF}} \sum_{aF} \phi_c(R).T - \sum_{aF} \phi_c(R).T \\ &= (\eta_{\substack{neutron \\ source}} - 1) \sum_{aF} \phi_c(R).T \end{aligned}$$

where: $\phi_c(R)$ is the value of the flux at the core and reflector interface where the neutron source is introduced.

We can write diffusion equations for the core and reflector regions as:

$$\begin{aligned} &\text{Core:} && D_c \nabla^2 \phi_c - \sum_{ac} \phi_c + \eta_c \sum_{aFc} \phi_c = 0 \\ &\text{Reflector:} && D_r \nabla^2 \phi_r - \sum_{ar} \phi_r = 0 \end{aligned}$$

Now we can suggest that for a fast unmoderated reactor material in the core with unity resonance escape probability p and fast fission factor ε:

$$k_\infty = \eta_c \varepsilon p f \simeq \eta_c .1.1. f = \eta_c f$$

and:

$$\eta_c \sum_{aF} = \eta_c \frac{\sum_{aF}}{\sum_{ac}} \sum_{ac} = \eta_c f \sum_{ac} = k_\infty \sum_{ac}$$

Substituting in the core and reflector diffusion theory equations:

$$\begin{aligned} &\nabla^2 \phi_c + \frac{(k_\infty - 1)}{L_c^2} \phi_c = 0, \; L_c^2 = \frac{D_c}{\sum_{ac}} \\ &\nabla^2 \phi_r - \frac{1}{L_r^2} \phi_r = 0, \; L_r^2 = \frac{D_r}{\sum_{ar}} \end{aligned}$$

If the material of the core is chosen with an infinite medium multiplication factor of unity, the material buckling in the core is:

$$B_c^2 = \frac{k_\infty - 1}{L_c^2} = \frac{1-1}{L_c^2} = 0$$

The zero material buckling of the core means that a flat flux distribution exists in the core implying a uniform power distribution, a desirable feature leading to uniform fuel burnup as well as heat generation.

18. SPHERICAL CORE

In spherical geometry:

$$\nabla^2 \phi_c(r) = 0,$$
$$\frac{1}{r^2}\frac{d}{dr}(r^2 \frac{d\phi_c(r)}{dr}) = 0,$$
$$\frac{d}{dr}(r^2 \frac{d\phi_c(r)}{dr}) = 0,$$
$$\int d(r^2 \frac{d\phi_c(r)}{dr}) = \int 0 dr,$$
$$(r^2 \frac{d\phi_c(r)}{dr}) = C,$$
$$\frac{d\phi_c(r)}{dr} = \frac{C}{r^2}, \forall r \neq 0,$$
$$\int d\phi_c(r) = \int \frac{C}{r^2} dr$$
$$\phi_c(r) = -\frac{C}{r} + F$$

For a finite flux in the core, C = 0, and:

$$\phi_c(r) = F = \text{constant}.$$

19. SEMI INFINITE SLAB GEOMETRY

In slab cartesian geometry, for a semi-infinite slab, this reduces to:

$$\nabla^2\phi_c(\mathrm{x}) = 0,$$
$$\frac{d^2\phi_c(\mathrm{x})}{dx^2}) = 0,$$
$$\frac{d}{dx}(\frac{d\phi_c(\mathrm{x})}{dx}) = 0,$$
$$\int d(\frac{d\phi_c(\mathrm{x})}{dx}) = \int 0dx,$$
$$\frac{d\phi_c(\mathrm{x})}{dx} = \mathrm{C},$$
$$\int d\phi_c(\mathrm{x}) = \int Cdx$$
$$\phi_c(\mathrm{x}) = Cx + F$$

For a finite flux in the core, C = 0, and again:

$$\phi_c(\mathrm{x}) = F = \text{constant}.$$

20. SEMI INFINITE CYLINDER GEOMETRY

In cylindrical geometry, for a semi-infinite cylinder, this reduces to:

$$\nabla^2\phi_c(r) = 0,$$
$$\frac{1}{r}\frac{d}{dr}(r\frac{d\phi_c(r)}{dr}) = 0,$$
$$\int d(r\frac{d\phi_c(r)}{dr}) = \int r.0dr,$$
$$r\frac{d\phi_c(r)}{dr} = C,$$
$$\frac{d\phi_c(r)}{dr} = \frac{C}{r}, \forall r \neq 0,$$
$$\int d\phi_c(r) = \int \frac{C}{r}dr$$
$$\phi_c(r) = C\ln r + F$$

For a finite flux in the core, C = 0, and again:

$$\phi_c(r) = F = \text{constant}.$$

The flux solution in the infinite reflector is:

$$\phi_r(r) = A\frac{e^{-\frac{r}{L_r}}}{r} + G\frac{e^{+\frac{r}{L_r}}}{r}$$

For a finite flux, G = 0 and:

$$\phi_r(r) = A\frac{e^{-\frac{r}{L_r}}}{r}$$

We can now apply the thin interface boundary conditions since the neutron source shell is considered as thin. The continuity of the flux and current at the interface yields:

$$\phi_c(R) = \phi_r(R)$$
$$J_{cn}(R) = J_{rn}(R) + J_{\substack{neutron \\ source}}$$

The flux continuity at the boundary implies:

$$F = A\frac{e^{-\frac{R}{L_r}}}{R}$$

The current boundary condition becomes:

$$D_c\nabla\phi_c(R) = D_r\nabla\phi_r(R) + J_{\substack{neutron \\ source}}$$

$$0 = A\frac{D_r}{R}[-\frac{e^{-\frac{R}{L_r}}}{R^2} - \frac{1}{L_r R}e^{-\frac{R}{L_r}}] + (\eta_{\substack{neutron \\ source}} - 1)\sum_{aF} A\frac{e^{-\frac{R}{L_r}}}{R}T$$

The constant A cancels out yielding the critical condition for the assembly as:

$$D_r[\frac{1}{R} + \frac{1}{L_r}] = T(\eta_{\substack{neutron \\ source}} - 1)\sum_{aF}$$

EXERCISES

1. You are given a bare spherical fast reactor of pure fissile material.
a) By equating the geometrical buckling to the material buckling, derive expressions for:
1) The critical radius
2) The critical volume
3) The critical mass.
b) Calculate these values for a U^{235} spherical reactor with:

microscopic transport cross section = 8.246 [barns]
microscopic absorption cross section = 2.844 [barns]
density = 18.75 [gm/cm^3]
product of average number of neutrons released in fission(ν) and the microscopic fission cross section = 5.297 [neutrons.barn].
Compare your result to the actual critical mass of the Godiva Experiment composed of 93.9 percent enriched uranium^{235} where $M_{critical}$= 48.8 kgs.
c) Calculate these values for a Pu^{239} spherical reactor with:
microscopic transport cross section = 6.8 barns
microscopic radiative capture cross section = 0.26 barns
density = 19.74 [gm/cm^3]
average number of neutrons released per fission ν = 2.98
microscopic fission cross section = 1.85 barns
Compare your result to the actual critical mass of the Jezebel Experiment composed of pure Pu^{239} where M(critical)= 20.53 kgs.
d) Discuss the criticality situation for a sphere made out of U^{238} with the following data:
microscopic transport cross section = 6.9 barns
microscopic radiative capture cross section = 0.16 barns
density = 19.05 [gm/cm^3]
average number of neutrons released per fission ν = 2.6
microscopic fission cross section = 0.095 barns.
Discuss the results of your calculations for the attainable critical masses for these materials.

2. Derive the expression for the flux distribution in a spherical reactor of radius R in terms of its power level P.

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