TRANSPORT THEORY

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1. INTRODUCTION

The term "Transport Theory" is used to refer to the mathematical description of the transport of particles, whether they are photons of electromagnetic radiation including light photons, x-rays or gamma rays, gas molecules, cars in traffic, neutrons, or charged particles such as electrons and protons, through a host medium. Some examples of transport processes are:

1. Neutron distributions in nuclear reactors,
2. Shielding of radioactive sources,
3. Propagation of light through stellar matter,
4. Penetration of light through the atmosphere.
5. Traffic flow on highways,
6. Gas dynamics,
7. Scattering of radar waves from the atmospheric turbulence,
8. Configuration of macromolecules,
9. Plasma dynamics.

We derive a general form of the Transport Equation and show its use in different fields of science, with an emphasis on neutron and gamma ray transport.

2. PARTICLES DENSITY

In particle transport, the random nature of particles motion allows us to use a field of probability density functions or distribution functions, rather than continuum descriptions such as electric and magnetic fields, local temperature, charge and current densities, mass density or local flow velocity.

The "Expected Particle Density" is defined as:

$$N(\underline{r},t)d^3r = \text{expected number of particles in } d^3r \text{ about } \underline{r} \text{ at time t.}$$

This particle density can then be described by an appropriate differential, integral, or integro-differential equation.

3. PARTICLES DISTRIBUTION FUNCTIONS

A classical point particle can be characterized by specifying the particle's position $\underline{r}$ and velocity $\underline{v}$. Thus we define a "particle phase space density" as:

$n(\underline{r},\underline{\text{v}},t)d^3rd^3v$ = expected number of particles in d^3r about $\underline{r}$ with velocity in d^3v about $\underline{\text{v}}$ at time t.

Integrating over the velocity variable:

$$N(\underline{r},t) = \int n(\underline{r},\underline{\text{v}},t)d^3\text{v} \tag{1}$$

If the particles are in thermal equilibrium at an absolute temperature T, then $n(\underline{r},\underline{\text{v}},t)$ becomes the Maxwell-Boltzmann distribution:

$$n(\underline{r},\underline{\text{v}},t) \rightarrow n_0 M(\underline{\text{v}}) = n_0 (\frac{m}{2\pi kT})^{3/2} e^{-\frac{m\text{v}^2}{2kT}} \tag{2}$$

where n_0 is the average number density of the particles.

It is common to normalize $n(\underline{r},\underline{\text{v}},t)$ as follows:

$$f(\underline{r},\underline{\text{v}},t) = \frac{n(\underline{r},\underline{\text{v}},t)}{N(\underline{r},t)} = \frac{n(\underline{r},\underline{\text{v}},t)}{\int n(\underline{r},\underline{\text{v}},t)d^3\text{v}} \tag{3}$$

which is a probability density function or distribution function that is normalized to unity:

$$\int f(\underline{r},\underline{\text{v}},t)d^3\text{v} = 1.$$

4. ANGULAR PARTICLES DENSITY

To specify the direction of particles motion we use a vector $\underline{\Omega}$ in the direction of the velocity vector $\underline{\text{v}}$, as shown in Fig. 1:

$$\underline{\Omega} = \frac{\underline{\text{v}}}{|\underline{\text{v}}|} = \sin\theta\cos\phi\ \hat{i} + \sin\theta\sin\phi\ \hat{j} + \cos\theta\ \hat{k} \tag{4}$$

The particle phase space density can thus be defined as:

$n(\underline{r},E,\underline{\Omega},t)d^3r\,dE\,d\underline{\Omega}$ = expected number of particles in d^3r about $\underline{r}$ with kinetic energy E in dE moving in the direction $\underline{\Omega}$ within the solid angle $d\underline{\Omega}$.

The differential solid angle is given by:

$$d\underline{\Omega} = \frac{dS}{r^2} = \frac{rd\theta . r\sin\theta d\phi}{r^2} = \sin\theta \, d\theta \, d\phi \qquad (5)$$

Integrating over the velocity space variable:

$$\int n(\underline{r},\underline{v},t)d^3v \; = \; \iint n(\underline{r},E,\underline{\Omega},t) \; dE \; d\Omega \; = \int_0^{\pi}\int_0^{2\pi}\int_0^{\infty} n(\underline{r},v,\underline{\Omega},t) \; \sin\theta d\theta d\phi \; mvdv \qquad (6)$$

where $dE = mv\,dv$, since $E = mv^2/2$

This is also called the "angular density" since it depends on θ and ϕ.

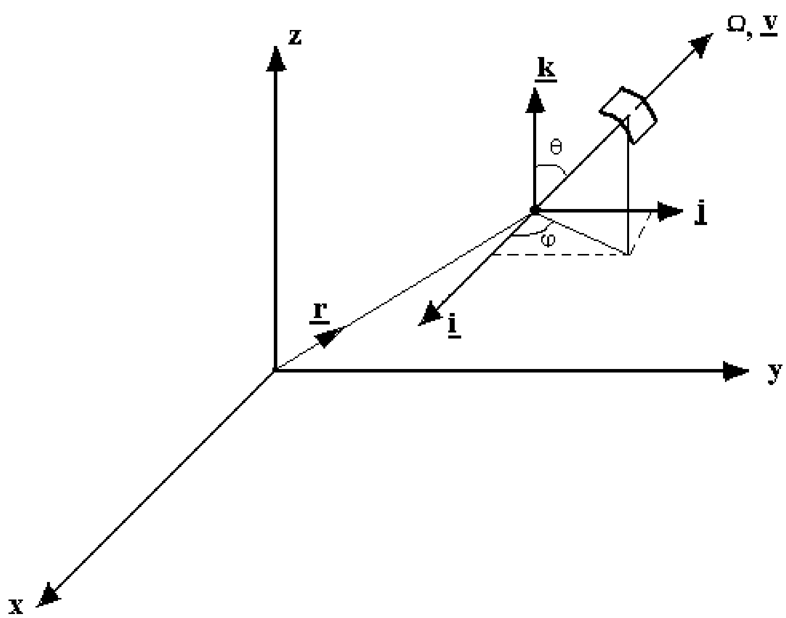

Fig. 1 Position and direction variables of a particle in spherical coordinates

5. CURRENT DENSITIES

The "phase space current density" function or "angular density" is defined as:

$\underline{j}(\underline{r},\underline{v},t)\,dS\,d^3v = \underline{v}\,n(\underline{r},\underline{v},t)\,dS\,d^3v =$ expected number of particles that cross an area $d\underline{S}$ per second with velocity $\underline{v}$ in d^3v at time t.

Integration over the particle velocities, yields the particle "current density" shown in Fig. 2 as:

$$\underline{J}(\underline{r},t) = \int \underline{j}(\underline{r},\underline{\mathrm{v}},t)\ d^3\mathrm{v} \tag{7}$$

This is referred to, as "flux" is fields other than neutron transport, since:

$\underline{J}(\underline{r},t)\ dS$ = net rate at which particles pass through a surface area dS.

It is a vector quantity characterizing the net rate at which particles pass through a surface oriented in a given direction.

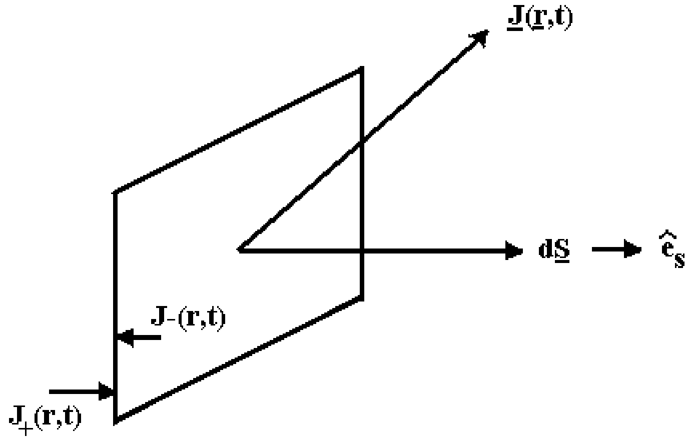

Fig 2 Relationship Between partial and total current densities

The "partial current density" characterizes the rate at which particles flow through the area in a given direction:

$$J_{\pm}(\underline{r},t) = \pm\int_{\pm} \hat{e}_s . \underline{j}(\underline{r},\underline{\mathrm{v}},t)\, d^3\mathrm{v} \tag{8}$$

where $\hat{e}_s$ is the unit normal to the surface, and the velocity integration is taken over those particles moving only in the positive or negative directions.

We can write:

$$\hat{e}_s . \underline{J}(\underline{r},t) = J_{+}(\underline{r},t) - J_{-}(\underline{r},t) \tag{9}$$

Thus $\underline{J}(\underline{r},t)$ is a "net current density," since it can be constructed as the difference of the partial current densities.

6. GENERIC FORM OF THE TRANSPORT EQUATION

We balance the various mechanisms by which particles can be gained or lost from an arbitrary volume V of material with surface S according to the geometry of Fig. 3:

{Time rate of change of n} = {Change due to leakage through surface S}
+{Change due to collisions}
+{Sources}

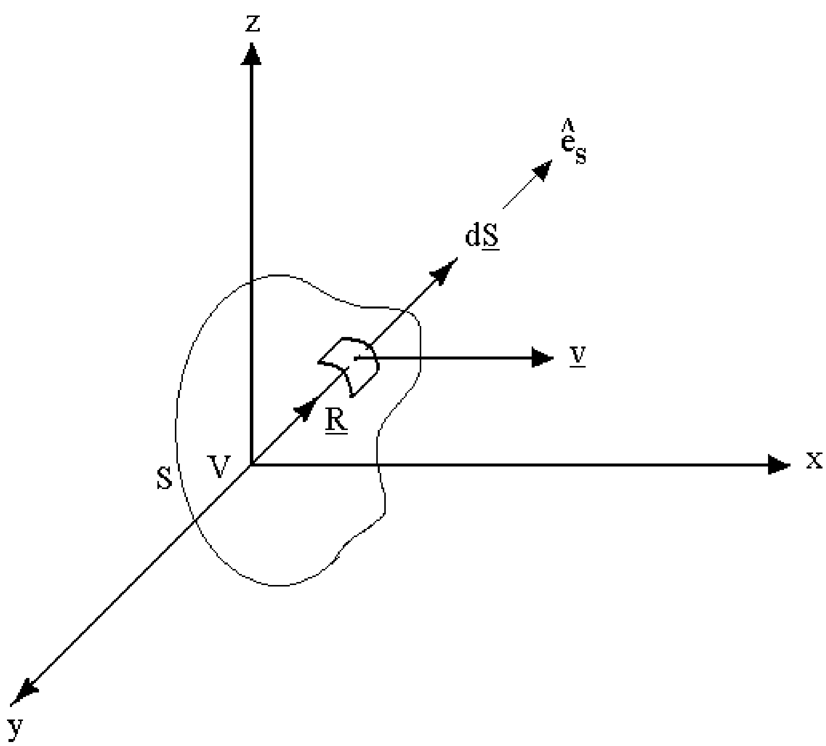

Fig. 3 - Arbitrary Volume V with surface area S

Mathematically this is expressed as follows:

$$\frac{\partial}{\partial t}\int_V n(\bar{r},\bar{\mathrm{v}},t).d^3r.d^3\mathrm{v} = -\int_S dS.\bar{j}(\bar{r},\bar{\mathrm{v}},t)d^3\mathrm{v}$$

$$+\int_V (\frac{\partial n}{\partial t})_{collision}.d^3r d^3\text{v}$$
$$+\int_V s(\overline{r},\overline{\text{v}},t)\,d^3r\,d^3\text{v} \qquad (10)$$

where $s(\overline{r},\overline{\text{v}},t)$ is a source density function.

If the choice of the arbitrary volume does not depend on time, then:

$$\frac{\partial}{\partial t}\int_V n(\overline{r},\overline{\text{v}},t)\,d^3r\,d^3\text{v} = \int_V \frac{\partial}{\partial t} n(\overline{r},\overline{\text{v}},t)\,d^3r\,d^3\text{v}$$

We can also use Gauss' Theorem or the Divergence Theorem to convert the surface integral into a volume integral:

$$\int_S dS.\overline{j}(\overline{r},\overline{\text{v}},t) = \int_V \nabla.\overline{j}(\overline{r},\overline{\text{v}},t)d^3r = \int_V \nabla.\overline{\text{v}}\ n(\overline{r},\overline{\text{v}},t)d^3r = \int_V \overline{\text{v}}.\nabla n(\overline{r},\overline{\text{v}},t)d^3r$$

since $\overline{r}$ and $\overline{\text{v}}$ are independent variables, and thus we can write:

$$\nabla.\overline{\text{v}}\,n(\overline{r},\overline{\text{v}},t)d^3r = \overline{\text{v}}.\nabla n(\overline{r},\overline{\text{v}},t)d^3r$$

The balance condition thus becomes:

$$\int_V [\frac{\partial n}{\partial t} + \overline{\text{v}}.\nabla n(\overline{r},\overline{\text{v}},t) - (\frac{\partial n}{\partial t})_{Collision} - s(\overline{r},\overline{\text{v}},t)]\ d^3r\,d^3\text{v} = 0 \qquad (11)$$

Since the volume V is arbitrary, the integrand itself is zero, thus:

$$\frac{\partial n}{\partial t} + \overline{\text{v}}.\nabla n(\overline{r},\overline{\text{v}},t) = (\frac{\partial n}{\partial t})_{Collision} + s(\overline{r},\overline{\text{v}},t) \qquad (12)$$

We can generalize the equation by using the “substantial derivative” relating the time rate of change of the local particle density along the particle trajectory to the change in the local density due to local collisions and sources:

$$(\frac{\partial n}{\partial t})_{Collision} + s(\bar{r},\bar{\mathrm{v}},t) = \frac{Dn(\bar{r},\bar{\mathrm{v}},t)}{Dt}$$

$$= \frac{\partial n}{\partial t} + \frac{\partial \bar{r}}{\partial t}.\frac{\partial n}{\partial \bar{r}} + \frac{\partial \bar{\mathrm{v}}}{\partial t}.\frac{\partial n}{\partial \bar{\mathrm{v}}}$$

$$= \frac{\partial n}{\partial t} + \underline{v}.\frac{\partial n}{\partial \bar{r}} + \frac{\bar{F}}{m}.\frac{\partial n}{\partial \bar{\mathrm{v}}}$$

$$where: \frac{\partial n}{\partial \bar{r}} = \nabla n = grad\ n$$

Thus the transport equation takes the form:

$$\frac{\partial n}{\partial t} + \bar{\mathrm{v}}.\frac{\partial n}{\partial \bar{r}} + \frac{\bar{F}}{m}.\frac{\partial n}{\partial \bar{\mathrm{v}}} = (\frac{\partial n}{\partial t})_{Collision} + s(\bar{r},\bar{\mathrm{v}},t) \quad (13)$$

Defining the macroscopic cross section for local interaction events:

$$\Sigma(\underline{r},\underline{v}) = N(\underline{r}).\sigma(\underline{v}) \quad (14)$$

where: $N(\underline{r})$ is the number density for the background medium,

$\sigma(\underline{v})$ is the microscopic interaction cross section.

The "collision frequency" can be written as

$v.\Sigma(\underline{r},\underline{v})$,[cm/sec].[cm^{-1}] = [1/sec]

The rate at which reactions occur per unit volume can be written as:

Reaction rate density = $v.\Sigma(\underline{r},\underline{v}) .n(\underline{r},\underline{v},t)$,

and its units are:

[cm/sec][cm^{-1}][particles/cm^3] = [interactions/(cm^3.sec)].

To describe the scattering process, we define the "collision kernel": $\Sigma(\underline{r},\underline{v}'\rightarrow \underline{v})$ as:

$$\Sigma(\underline{r},\underline{v}'\rightarrow \underline{v}) = \Sigma(\underline{r},\underline{v}')\, c(\underline{r},\underline{v}')\, f(\underline{r},\underline{v}'\rightarrow \underline{v}) \quad (15)$$

where $f(\underline{r},\underline{v}'\rightarrow \underline{v})$ = probability that any secondary particle induced by an incident particle with velocity $\underline{v}'$ will be emitted with velocity $\underline{v}$ in d^3v at $\underline{r}$.

$c(\underline{r},\underline{v}')$ = mean number of secondary particles induced by an incident particle with velocity $\underline{v}'$ will be emitted with velocity $\underline{v}$ in d^3v at $\underline{r}$.

The collision term can thus be written as:

$$\left(\frac{\partial n}{\partial t}\right)_{Collision} = \left\{\begin{array}{l}\text{rate at which particles of velocity } \underline{v}' \text{ induce the production} \\ \text{of secondary particles of velocity } \underline{v}\end{array}\right\}$$

$$-\left\{\begin{array}{l}\text{rate of interactions for particles of velocity } \underline{v} \text{ that change their} \\ \text{velocity or destroy the particle}\end{array}\right\}$$

Thus:

$$\left(\frac{\partial n}{\partial t}\right)_{Collision} = \int v' \, \Sigma(\underline{r}, \underline{v}' \longrightarrow \underline{v}) \, n(\underline{r}, \underline{v}', t) \, d^3v' - v \, \Sigma(\underline{r}, \underline{v}) \, n(\underline{r}, \underline{v}', t) \qquad (16)$$

And the general form of the Transport Equation becomes:

$$\frac{\partial n}{\partial t} + \underline{v}.\frac{\partial n}{\partial \underline{r}} + \frac{\underline{F}}{m}.\frac{\partial n}{\partial \underline{v}} + v \Sigma n = \int v' \, \Sigma(\underline{r}, \underline{v}' \longrightarrow \underline{v}) \, n(\underline{r}, \underline{v}', t) \, d^3v' + s \qquad (17)$$

7. THE ANGULAR FLUX, OR PHASE SPACE FLUX

The product $v.n(\underline{r},\underline{v},t)$ arises so frequently that it was given a special notation:

$\varphi(\underline{r},\underline{v},t) = v.n(\underline{r},\underline{v},t)$ = angular flux.

The velocity-integrated flux is given by:

$$\varphi(\underline{r},t) = \int \varphi(\underline{r},\underline{v},t) \, d^3v = \int v \, n(\underline{r},\underline{v},t) \, d^3v \qquad (18)$$

The tradition in nuclear applications of calling this quantity "flux" could be misleading. It is here a scalar quantity whereas other fluxes met-with in mathematical physics are vector quantities. Actually, the current density $\underline{J}(\underline{r},t)$ is closely related to the conventional definition of the flux.

The units of both $\underline{J}(\underline{r},t)$ and $\varphi(\underline{r},t)$ are identical: [particles/(cm^2 sec)]. However, $\underline{J}(\underline{r},t)$ is a vector quantity characterizing the net rate at which particles pass through a surface oriented in a given direction, whereas $\varphi(\underline{r},t)$ characterizes the total rate at which particles pass through a unit area, regardless of orientation.

Thus $\underline{J}(\underline{r},t)$ is an appropriate quantity to estimate *leakage or flow*. $\varphi(\underline{r},t)$ is more suitable for estimating *reaction rates*.

The relationship between the angular flux and angular current density is:

$$\underline{j}(\underline{r},\underline{v},t) = \underline{\Omega}.\varphi(\underline{r},\underline{v},t) \qquad (19)$$

In general, there is no direct relationship between $\varphi(\underline{r},t)$ and $\underline{J}(\underline{r},t)$ since they are quite different moments of the particle distribution function:

$$\varphi(\underline{r},t) = \int \mathrm{v}\, n(\underline{r},\underline{\mathrm{v}},t) d^3\mathrm{v} ,$$

$$\underline{J}(\underline{r},t) = \int \underline{\mathrm{v}}\, n(\underline{r},\underline{\mathrm{v}},t) d^3\mathrm{v} .$$

If we ignore the external force $\underline{F}$, we can write the integro-differential form of the Transport Equation in terms of the flux as:

$$\frac{1}{v}\frac{\partial\varphi}{\partial t} + \underline{\Omega}.\nabla\varphi + \Sigma\varphi = \int_0^\infty \int_{4\pi} \Sigma(E' \longrightarrow E, \Omega' \longrightarrow \Omega)\ \varphi(\underline{r}, E', \underline{\Omega}', t)\, dE'\, d\underline{\Omega}' + s \quad (20)$$

8. THE ONE-DIMENSIONAL FORM OF THE NEUTRON TRANSPORT EQUATION

In Cartesian coordinates and one-dimension, the streaming term becomes

$$\underline{\Omega}.\nabla\varphi(\mathrm{x}) = (\Omega_x \frac{\partial}{\partial \mathrm{x}} + \Omega_y \frac{\partial}{\partial \mathrm{y}} + \Omega_z \frac{\partial}{\partial \mathrm{z}})\varphi(x) = \Omega_x \frac{\partial}{\partial \mathrm{x}}\varphi(x)$$

Let:

$$\Omega_x = \cos\theta = \mu$$

and consider no dependence on the azimuthal angle φ. Thus we can write the one-dimensional transport equation in cartesian coordinates as:

$$\frac{1}{v}\frac{\partial\varphi}{\partial t} + \mu\frac{\partial\varphi}{\partial x} + \Sigma\varphi = \int_0^\infty \int_{-1}^{+1} \Sigma(E' \longrightarrow E, \mu' \longrightarrow \mu)\ \varphi(x, E', \mu', t)\, dE'\, d\mu' + s \quad (21)$$

9. THE INITIAL AND BOUNDARY CONDITIONS TO THE TRANSPORT EQUATION

Since a first order time derivative appears in the equation, the initial condition needed is:

$$n(\underline{r} = \underline{R}, \underline{\mathrm{v}}, t = 0) = n_0(\underline{r}, \underline{\mathrm{v}}), \text{ for all } \underline{r} \text{ and } \underline{\mathrm{v}}$$

The commonly used boundary conditions are of three types:

1. **Free surface boundary condition**:

In this case, particles can only escape a body through the surface, but cannot reenter it. Thus the density must vanish on the surface for all inward directions:

$$n(\underline{r} = \underline{R}, \underline{v}, t) = 0 , \quad \text{for all } \underline{v}.\hat{e}_s < 0$$

2. **Reflecting boundary conditions**:

The incoming density is reduced in this case by other albedo factor α:

$$n(\underline{r} = \underline{R}, \underline{v}, t) = \alpha .\, n(\underline{r} = \underline{R}, \underline{v}, t), \quad \text{for all } \underline{v}.\hat{e}_s < 0$$

Such that:

$$\underline{v}.\hat{e}_s = \underline{v_r}.\hat{e}_s$$

and:

$$(\underline{v} x \underline{v_r}).\hat{e}_s = 0 \qquad or \qquad v\, x\, \hat{e}_s = \underline{v_r}\, x\, \hat{e}_s$$

where α =1, this leads to "specular reflection" of the particles without loss.

3. **Periodic Boundary Conditions**:

The outgoing density on certain boundaries is equated with the incoming density on other boundaries that are related by symmetric conditions. This leads to the identification of computational unit cells.

4. **Interface Boundary Conditions**:

Since nothing of infinitesimal thickness can create or destroy particles we can write at the interfaces:

$$n(\underline{r} = \underline{R_1}, \underline{v}, t) = n(\underline{r} = \underline{R_2}, \underline{v}, t) , \quad \text{for all } \underline{v}.$$

5. **Infinity Boundary Condition**:

The density should be well behaved at infinity:

$$\lim_{|\underline{r}| \longrightarrow \infty} n(\underline{r}, \underline{v}, t) < \infty$$

10. **APPROXIMATIONS TO THE TRANSPORT EQUATION**:

Seven independent variables: x, y, z, v_x, v_y, v_z and t, are involved in the solution of the Transport Equation. Moreover, the dependence of the collision cross section $\Sigma(\underline{r},\underline{v}'\rightarrow\underline{v})$ on particle velocity v is extremely complicated because of the collision dynamics. No computer is sufficiently large to solve the general Transport Equation. Three ways are available to consider transport problems:

1. Approximating the form of the equation itself e.g. Diffusion Theory.
2. Consideration of model problems for which the form of the Transport Equation is simple enough.
3. Use of numerical or statistical simulation techniques such as the Monte Carlo method.

Approximations to the geometry, energy dependence or angular dependence are normally introduced to make it possible to solve the Transport Equation:

1. **Geometrical Approximations**:
a. Isotropic, homogeneous media,
b. Infinite media or half-spaces,
c. One dimensional plane, spherical, or cylindrical geometry,
d. Periodic lattice symmetry.

2. **Energy Approximations**:
a. One-speed approximation,
b. Multigroup energy descriptions,
c. Expansion of the energy dependence in polynomial,
d. Simple models of the collision kernels.

3. **Angular dependence approximations**:
a. Isotropic sources,
b. Isotropic scattering,
c. Expansion of collision kernels in Legendre Polynomials in angle.

11. THE NEUTRON TRANSPORT EQUATION:

The neutron phase space density is taken as $n(\underline{r},\underline{v},\underline{t})$ and $\Sigma(\underline{r},\underline{v})$ is taken as the microscopic cross-section characterizing neutron-nuclear interactions. The Transport Equation then takes the form:

$$\frac{1}{v}\frac{\partial\varphi}{\partial t} + \underline{\Omega}.\nabla\varphi + \Sigma_t\,\varphi = \int_0^\infty\int_{4\pi}\Sigma(E'\longrightarrow E,\Omega'\longrightarrow\Omega)\;\varphi(\underline{r},E',\underline{\Omega}',t)\,dE'\,d\underline{\Omega}' + s \quad (22)$$

The initial and boundary conditions are:

$$\varphi(\underline{r},E,\underline{\Omega},0) = \varphi(\underline{r},E,\underline{\Omega})$$

$$\varphi(\underline{R},E,\underline{\Omega},t) = 0 \text{ for all } \underline{v}.\hat{e}_s < 0 \;(\textit{free surface})$$

In fission reactor systems, the source term takes the form:

$$s_f(\underline{r},E,\underline{\Omega},t) = \frac{x(E)}{4\pi}\int_0^{\infty}\int_{4\pi}\upsilon(E')\Sigma_f(E')\ \varphi(\underline{r},E',\underline{\Omega}',t)\,dE'\,d\underline{\Omega}' \qquad (23)$$

where ν(E') is the average number of neutrons released per fission,

χ(E') is the energy distribution or spectrum of the fission neutrons.

The fission spectrum is given by:

$$\chi\,(\mathrm{E}) = 0.453e^{-1\,036E}\ \sinh\sqrt{2.29E} \qquad (24)$$

Three categories of problems present themselves in the solution of this equation:

1.**Source Problems**:
a. Sources in infinite media,
b. Behavior of flux near a free surface or the Milne problem,
c. Albedo problems,
d. Finite geometry problems.

2. **Criticality problems**:
These involve the system composition and/or geometry such that the fission neutrons production just balance neutron absorption and leakage to yield a time-independent solution to the Transport Equation.

3. **Time-dependent problem**:
a. Pulsed- neutron problems: initial value problems,
b. Neutron Wave problems: response to time-modulated sources.

12. THE PHOTON TRANSPORT OR RADIATIVE TRANSFER EQUATION:

To describe the transport of low energy photons, the photon energy intensity is defined in terms of the photon flux φ(r,v,t) and photon energy (hν) as :

$$I(\underline{r},\underline{\Omega},t) = (h\nu)\ \varphi(\underline{r},\underline{v},t) \qquad (25)$$

The Radiative Transport Equation becomes:

$$\frac{1}{c}\frac{\partial I}{\partial t} + \Omega.\nabla I = \rho(\underline{r},t)\left[\varepsilon(\underline{r},\underline{\Omega},t) - K(\underline{r},\underline{\Omega},t)\,I(\underline{r},\underline{\Omega},t)\right]$$

where $\rho(\underline{r},t)$ *is the local matter density,* (26)

$K(\underline{r},\underline{\Omega},t)$ *is the absorption coefficient,*

$\varepsilon(\underline{r},\underline{\Omega},t)$ *is the emission coefficient.*

This can be simplified for local thermodynamic equilibrium as:

$$\frac{1}{c}\frac{\partial I}{\partial t} + \Omega.\nabla I = \rho K[S - I] \qquad (27)$$

where the emission term is given by:

$$S = \frac{2h\upsilon^3}{c^2}[e^{\frac{h\upsilon}{kT}} - 1]^{-1}$$

where h is Planck’s constant.

Some example problems are:
a. High energy gamma transport, deep penetration and shielding,
b. Stellar atmosphere problems: the Milne problem,
c. Radiation penetration into stellar atmosphere: Albedo problem,
d. Radiative transfer in plasmas.

13. **HIGH ENERGY, CHARGED PARTICLE TRANSPORT EQUATION**:

Two processes characterize this phenomenon:
a. The strong but infrequent collisions of the particles with heavy ions, with little energy transfer.
b. Frequent weak collisions with atomic electrons, giving rise to very irregular trajectories.

One starts by using an energy-independent description to account for elastic collisions with heavy ions:

$$\frac{1}{v}\frac{\partial \varphi}{\partial t} + \underline{\Omega}.\nabla\varphi + \Sigma_s\,\varphi = \int \Sigma_s(\Omega' \longrightarrow \Omega)\ \varphi(\underline{r}, v, \underline{\Omega}', t)\ d\underline{\Omega}' + s \qquad (28)$$

The frequent, weak collisions are introduced through a continuous slowing down theory. In this case one knows the energy loss over a given path length ξ: dE/dξ.
Using the relationship:

$$d\xi = vdt \qquad (29)$$

The independent variables in the last equation are transformed to yield:

$$\frac{\partial \varphi}{\partial \xi} + \underline{\Omega}.\nabla\varphi + \Sigma_s\,\varphi = \int \Sigma_s(\Omega' \longrightarrow \Omega)\ \varphi(\underline{\Omega}')\ d\underline{\Omega}' + s \qquad (30)$$

Examples of encountered problems are:
a. Shielding against charged particle radiation on space missions,
b. Electron plasma production using electron beams.

14. THE BOLTZMANN TRANSPORT EQUATION FOR GAS DYNAMICS:

Here we have a motion of gas molecules colliding with each other:

$$\frac{\partial n}{\partial t} + \underline{v}.\nabla n = \iint |\underline{v}_1 - \underline{v}|\, \sigma(\underline{\Omega}, |\underline{v}_1 - \underline{v}|).[n(\underline{v}'_1)n(\underline{v}) - n(\underline{v}'_1)n(\underline{v})]\, d^3v\, d\underline{\Omega} \quad (31)$$

The primes indicate the molecular velocities prior to the collision event and n($\underline{v}$) is a shorthand for n($\underline{v}$,$\underline{r}$,t). This equation contains a quadratic nonlinearity.

The equation can be linearized for small disturbances around the equilibrium distribution:

$$n_0(\underline{v}) = n_0 M(\underline{v}),$$

by defining:

$$n(\underline{r},\underline{v},t) = n_0(\underline{v}) + n_1(\underline{r},\underline{v},t) \qquad , |n_1| << |n_0|$$

If only first-order terms in the perturbation are retained after substitution into the Boltzmann Equation, we get the Linearized Boltzmann Equation in the perturbation n_1:

$$\frac{\partial n}{\partial t} + \underline{v}.\nabla n_1 = \iint |\underline{v}_1 - \underline{v}|\, \sigma\, n_0(\underline{v})\, [n_1\underline{v}'_1 + n_1\underline{v}' - n_1(\underline{v}_1) - n_1(\underline{v})]\, d^3v_1\, d\underline{\Omega} \quad (32)$$

If the term $n_0(\underline{v}).\sigma(\underline{\Omega},|\underline{v}_1 - \underline{v}|)$ is identified as the scattering kernel $\Sigma_s(\underline{v}' \rightarrow \underline{v})$, the equation takes the form of the Neutron Transport Equation.

Some examples of problems treated are:
1. Shock wave propagation,
2. Sound wave propagation,
3. Steady flow, gas-surface interaction, heat transfer.

15. IONIZED GASES AND PLASMAS TRANSPORT EQUATION:

The long-range nature of Coulomb Interaction must be accounted for. Using the Lorentz equation for an electric field $\underline{E}$ and magnetic field $\underline{B}$:

$$\underline{F} = q\,(\underline{E} + \underline{v} \times \underline{B}) \quad (33)$$

Thus, for electrons (q = -e) the Transport Equation becomes:

$$\frac{\partial n_e}{\partial t} + \underline{v}.\frac{\partial n_e}{\partial \underline{r}} - \frac{e}{m}(\underline{E} + \underline{v}\times\underline{B})\frac{\partial n_e}{\partial \underline{v}} = (\frac{\partial n_e}{\partial t})_{Collision} \qquad (34)$$

In a plasma, which is an ionized gas with a Debye length small compared to the system size, collective motions are very important, and E and B must be determined using Maxwell's equations.

For the study of electrostatic oscillations in the electron density, one ignores the collision term and writes:

$$\frac{\partial n_e}{\partial t} + \underline{v}.\frac{\partial n_e}{\partial \underline{r}} - \frac{e}{m}\underline{E}.\frac{\partial n_e}{\partial \underline{v}} = 0 \qquad (35)$$

where $\underline{E}(\underline{r},t)$ is determined by Poisson's Equation:

$$\frac{\partial E}{\partial \underline{r}} = -4\pi e \int [n_e(\underline{v}) - n_i(\underline{v})]\, d^3 v \qquad (36)$$

This equation, in which the collision term is ignored, and the electric field is explicitly accounted for, is known as Vlasov's equation and forms the basis of a majority of plasma physics applications.

Some of the problems treated are:

a. Propagation of shock waves in plasmas,
b. Plasma turbulence studies,
c. Wave propagation in plasmas,
d. Instabilities in $n(\underline{r},\underline{v},t)$, also called micro-instabilities.

16. DISCUSSION

The Transport equation encompasses many fields in science and engineering. A great deal of effort goes into solving it under different conditions and approximations. Some forms of the equation lends itself to different methods of approximation and solution.

The integro-differential form of the equation is suitable for finite difference and approximate analytical methods approaches, whereas its form as an integral equation lends itself to statistical integration methods such as Monte Carlo.