STATISTICAL WEIGHTING METHODS: TREATMENT OF SOURCE AND CRITICALITY MULTIPLYING SYSTEMS.

© M. Ragheb
3/17/2013

INTRODUCTION

In practical Monte Carlo calculations, Analog Monte Carlo is rarely used. Statistical weighting is used, particularly in conjunction with variance reduction methods. We here first consider the commonly used Absorption weighting method together with its application to criticality calculations using the source iteration method, or to source problems such as shielding or fusion blankets. The methods attributed to Ragheb and Maynard of Multiplication weighting, Generalized secondary weighting and Secondary weighting are introduced are next introduced. They are particularly useful for the efficient treatment source driven multiplying systems arising in hybrid fission-fusion blankets and accelerator breeding or waste transmutation studies.

ANALOG MONTE CARLO FOR THE TREATMENT OF MULTIPLYING SYSTEMS

When the neutron transport equation in multiplying media is to be solved by use of Monte Carlo, the most obvious approach is the analog simulation of the transport equation, choosing events according to their associated physical probabilities as shown in Table 1.

Table 1. Analog Monte Carlo.

Probabilities of reactions	Statistical weight adjustment at n-th collision	Simulation action
Scattering: $\frac{\Sigma_s}{\Sigma_t}$	$W_n=W_{n-1}$	Follow 1 scattered particle.
Capture: $\frac{\Sigma_c}{\Sigma_t}$	$W_n=0$	Terminate particle history.
Fission: $\frac{\Sigma_f}{\Sigma_t}$	$W_n=W_{n-1}$	Follow ν_f fission particles.
(n,2n): $\frac{\Sigma_{n,2n}}{\Sigma_t}$	$W_n=W_{n-1}$	Follow 2 scattered particles.
(n,3n): $\frac{\Sigma_{n,3n}}{\Sigma_t}$	$W_n=W_{n-1}$	Follow 3 scattered particles.

In this case the total macroscopic cross section is defined as:

$$\Sigma_t = \Sigma_s + \Sigma_c + \Sigma_f + \Sigma_{n,2n} + \Sigma_{n,3n} \tag{1}$$

Even in the absence of fission, this leads to a slow statistical convergence of the results. With fission present the problem would be untenable due to the need to follow an exponentially increasing number of particle tracks, leading to a combinatorial explosion. To avoid some of the difficulties, the statistical simulation can be modified to allow only secondary producing reactions with the effect of capture being accounted for by modifying the statistical weight of the particle being followed. The weight modification is determined by requiring that the weight expectation of the total number of secondary particles emerging from a collision is conserved.

GENERALIZED SECONDARY WEIGHTING METHOD

It must be first noticed that the probabilities of the different secondary producing reactions is arbitrary as far as the theoretical results are concerned, but the rate of convergence will vary with the choices made. They must, however add to unity. With such a method, a particle history would never terminate and it must be terminated by leakage from the system or by some weight related procedure.

Since many data libraries include in the matrices for energy and angle change the contributions from neutron producing reactions such as (n, 2n) and (n, 3n), the number of secondaries from scattering is not unity, but some value ν_s slightly greater than one given by:

$$\begin{aligned} \nu_s &= \frac{\Sigma_s + 2\Sigma_{n,2n} + 3\Sigma_{n,3n}}{\Sigma_s + \Sigma_{n,2n} + \Sigma_{n,3n}} \\ &= \frac{\Sigma_s + 2\Sigma_{n,2n} + 3\Sigma_{n,3n}}{\Sigma_{s'}} \end{aligned} \tag{2}$$

This treatment allows all secondary producing reactions to be treated as either scattering or fission.

The quantity:

$$\nu \frac{\Sigma'_s}{\Sigma_t}$$

is what is referred to in common Monte Carlo multigroup codes as the “nonabsorption probability.” This is a misnomer, since it is a probability only in the absence of the (n,2n) and (n,3n0 neutron multiplying processes.

Referring to Table 1, the analog simulation procedure is characterized by the physical probabilities of various reactions with the statistical weight of a particle kept at its original unit weight and the track continued until terminated by leakage or capture. All branches of all non fission particle producing reactions are stored , retrieved one a time and individually followed. The combination of all nonfisssion particle producing reactions into a generalized scattering reaction and the elimination of capture from consideration leads to the Generalized secondary Weighting Method shown in Table 2, where scattering occurs with probability P_s and fission with probability P_f subject only to the normalization condition:

$$P_s + P_f = 1 \tag{3}$$

Table 2. Generalized Secondary Weighting Method.

Probabilities of reactions	Statistical weight adjustment at n-th collision	Simulation action
Scattering: P_s	$W_n = \nu_s . \frac{\Sigma'_s}{\Sigma_t} . \frac{1}{P_s} . W_{n-1}$	Follow until weight related truncation
Fission: P_f (P_s+P_f=1)	$W_n = \nu_f . \frac{\Sigma_f}{\Sigma_t} . \frac{1}{P_f} . W_{n-1}$	Follow until weight related truncation

As will be shown later, the weighting preserves the total expected number of secondary particles, but in general does not conserve the number of reactions of a given type.

SPECIAL CASES OF STATISTICAL WEIGHTING METHOD: SECONDARY WEIGHTING, MULTIPLICATION WEIGHTING AND ABSORPTION WEIGHTING METHODS

The general form in the last section can be specialized to yield several interesting special cases including the Absorption Weighting method.

First, if Ps is chosen as the probability of scattering, given that only scattering and fission are allowed, the Secondary Weighting method results. As shown in Table 3, this creates the correct number of non-capture reactions of each type and the correct number of secondary particles of each type.

Table 3. Secondary Weighting Method.

Probabilities of reactions	Statistical weight adjustment at n-th collision	Simulation action
Scattering: $\frac{\Sigma_s}{(\Sigma_s + \Sigma_f)}$	$W_n = \nu_s . \frac{\Sigma'_s}{\Sigma_s} \frac{(\Sigma_s + \Sigma_f)}{\Sigma_t} . W_{n-1}$	Follow until weight related truncation
Fission: $\frac{\Sigma_f}{(\Sigma_s + \Sigma_f)}$	$W_n = \nu_f . \frac{(\Sigma_s + \Sigma_f)}{\Sigma_t} . W_{n-1}$	Follow until weight related truncation

Another choice is to assign the scattering probabilities on the basis of the number of secondary particles expected from each reaction as shown in Table 4. This is referred to as the Multiplication Weighting method since all particle weights are changed by the expected particle multiplication in the collision.

Table 4. Multiplication Weighting Method.

Probabilities of reactions	Statistical weight adjustment at n-th collision	Simulation action
Scattering: $\frac{\nu_s \Sigma'_s}{(\nu_s \Sigma'_s + \nu_f \Sigma_f)}$	$W_n = \frac{(\nu_s \Sigma'_s + \nu_f \Sigma_f)}{\Sigma_t} . W_{n-1}$	Follow until weight related truncation
Fission: $\frac{\nu_f \Sigma_f}{(\nu_s \Sigma'_s + \nu_f \Sigma_f)}$	$W_n = \frac{(\nu_s \Sigma'_s + \nu_f \Sigma_f)}{\Sigma_t} . W_{n-1}$	Follow until weight related truncation

Finally, by choosing unit scattering probability, the so-called absorption weighting method is obtained but must be coupled to additional procedures that store the generated fission particles, for later retrieval and following one at a time, as shown in Table 5. The weight change is the number of secondary scattered particles per collision, which in the case where ν_s is unity; when (n,2n) and (n,3n) reactions are not considered, is the non-absorption probability and accounts for the name. The weight of the particle is also changed by the factor:

$$\nu_f \frac{\Sigma_f}{\Sigma_t},$$

the number of fission secondaries per collision, and this result is stored for following later.

Table 5. Absorption Weighting Method.

Probabilities of reactions	Statistical weight adjustment at n-th collision	Simulation action
Scattering: 1	$W_n = \nu_s \frac{\Sigma'_s}{\Sigma_t} . W_{n-1}$	Follow until weight related truncation, <u>and</u> store at each collision as a fission source a weight: $W_n = \nu_f \cdot \frac{\Sigma_f}{\Sigma_t} W_{n-1}$
Fission: 0	-	-

COMPARISON OF DIFFERENT STATISTICAL WEIGHTING METHODS

In all cases, the total expected statistical weight $\bar{W}_n$ of particles emerging from a collision is preserved. Using the probabilities of reactions and the associated modified weights for the different processes from Table 1 for the analog process, multiplying the events probabilities by their associated particle weights, and summing:

$$\bar{W}_n = W_{n-1}(\frac{\Sigma_s}{\Sigma_t}.1 + \frac{\Sigma_f}{\Sigma_t}.\nu_f + \frac{\Sigma_c}{\Sigma_t}.0 + \frac{\Sigma_{n,2n}}{\Sigma_t}.2 + \frac{\Sigma_{n,3n}}{\Sigma_t}.3) \qquad (4)$$

Using Eqns 1 and 2 for the Generalized Secondary weighting, it is:

$$\bar{W}_n = W_{n-1}[P_s.(\nu_s.\frac{\Sigma'_s}{\Sigma_t}.\frac{1}{P_s}) + P_f.(\nu_f.\frac{\Sigma_f}{\Sigma_t}.\frac{1}{P_f})] \qquad (5)$$

For the Secondary Weighting method:

$$\bar{W}_n = W_{n-1}[(\frac{\Sigma_s}{\Sigma_s+\Sigma_f}.\{\frac{\nu_s\Sigma'_s}{\Sigma_s}.\frac{\Sigma_s+\Sigma_f}{\Sigma_t}\}) + (\frac{\Sigma_f}{\Sigma_s+\Sigma_f}.\{\nu_f.\frac{\Sigma_s+\Sigma_f}{\Sigma_t}\})], \qquad (6)$$

and for the Multiplication Weighting method:

$$\bar{W}_n = W_{n-1}[(\frac{\nu_s\Sigma'_s}{\nu_s\Sigma'_s+\nu_f\Sigma_f}.\frac{\nu_s\Sigma'_s+\nu_f\Sigma_f}{\Sigma_t}) + (\frac{\nu_f\Sigma_f}{\nu_s\Sigma'_s+\nu_f\Sigma_f}.\frac{\nu_s\Sigma'_s+\nu_f\Sigma_f}{\Sigma_t})] \qquad (7)$$

Since the quantities $\nu_s\Sigma'_s$ *and* $\nu_f\Sigma_f$ are readily available in scattering matrices for multigroup crosssections, Multiplication Weighting is easy to implement. Secondary weighting would require further cross section handling since it requires the knowledge of not only $\nu_s\Sigma'_s$ but also of Σ_f *and* Σ_s on an individual basis.

In Table 6, the information of the last sections is summarized and the relationship of the different statistical methods to Analog Monte Carlo is shown.

Table 6. Relationship of statistical weighting methods to Analog Monte Carlo.

Statistical weighting Method	Probabilities of reactions	Statistical weight adjustment at n-th collision	Simulation action
Analog	Scattering: $\frac{\Sigma_s}{\Sigma_t}$	$W_n=W_{n-1}$	Follow 1 scattered particle.

	Capture: $\frac{\Sigma_c}{\Sigma_t}$	$W_n=0$	Terminate particle history.
	Fission: $\frac{\Sigma_f}{\Sigma_t}$	$W_n=W_{n-1}$	Follow ν_f fission particles.
	(n,2n): $\frac{\Sigma_{n,2n}}{\Sigma_t}$	$W_n=W_{n-1}$	Follow 2 scattered particles.
	(n,3n): $\frac{\Sigma_{n,3n}}{\Sigma_t}$	$W_n=W_{n-1}$	Follow 3 scattered particles.
Generalized Secondary Weighting	Scattering: P_s	$W_n=\nu_s.\frac{\Sigma'_s}{\Sigma_t}.\frac{1}{P_s}.W_{n-1}$	Follow until weight related truncation
	Fission: P_f $(P_s+P_f=1)$	$W_n=\nu_f.\frac{\Sigma_f}{\Sigma_t}.\frac{1}{P_f}.W_{n-1}$	Follow until weight related truncation
Secondary Weighting	Scattering: $\frac{\Sigma_s}{(\Sigma_s+\Sigma_f)}$	$W_n=\nu_s.\frac{\Sigma'_s}{\Sigma_s}\frac{(\Sigma_s+\Sigma_f)}{\Sigma_t}.W_{n-1}$	Follow until weight related truncation
	Fission: $\frac{\Sigma_f}{(\Sigma_s+\Sigma_f)}$	$W_n=\nu_f.\frac{(\Sigma_s+\Sigma_f)}{\Sigma_t}.W_{n-1}$	Follow until weight related truncation
Multiplication Weighting	Scattering: $\frac{\nu_s\Sigma'_s}{(\nu_s\Sigma'_s+\nu_f\Sigma_f)}$	$W_n=\frac{(\nu_s\Sigma'_s+\nu_f\Sigma_f)}{\Sigma_t}.W_{n-1}$	Follow until weight related truncation
	Fission: $\frac{\nu_f\Sigma_f}{(\nu_s\Sigma'_s+\nu_f\Sigma_f)}$	$W_n=\frac{(\nu_s\Sigma'_s+\nu_f\Sigma_f)}{\Sigma_t}.W_{n-1}$	Follow until weight related truncation
Absorption Weighting	Scattering: 1	$W_n=\nu_s\frac{\Sigma'_s}{\Sigma_t}.W_{n-1}$	Follow until weight related truncation, and store at each collision as a fission source a weight: $W_n=\nu_f.\frac{\Sigma_f}{\Sigma_t}W_{n-1}$
	Fission: 0	-	-

THE SOURCE ITERATION METHOD WITH ABSORPTION WEIGHTING FOR CRITICALITY CALCULATIONS

The source iteration method allows the solution of criticality problems as a set of consecutive source problems to which the Monte Carlo techniques analyzed earlier are applicable.

The collision density (ingoing) can be written as:

$$\psi = S_c + T(C+F)\psi \quad (8)$$

where: S_c=TS is the uncollided neutron contribution to each phase space point from the external source S.

Here the original collision kernel C has added to it a fission kernel F which is the product of the macroscopic fission cross section Σ_f for the incident neutrons, the yield factor ν, and the energy and direction distribution function χ of the secondary neutrons as:

$$F(\bar{E}',\bar{E}\,|\,\bar{r}') = \nu(E')\Sigma_f(\bar{r}',E')\chi(\bar{E}) \quad (9)$$

Since the composite kernel T(C+F) will have generalyy a spectral radius very close to unity for multiplying media, convergence of the corresponding Neumann series of the solution will be slow, if unattainable for just critical systems. To get a general method which treats both source and eigenvalue problems, we split the kernel T(C+F) into two parts. The first part TC describes nonmultiplying particle transport, and the second part TF describes all neutron multiplying processes, thus:

$$\psi = S_c + (TC)\psi + (TF)\psi \quad (10)$$

The method of solving the last equation is now based on the physical analog of subsequent neutron generations. The neutrons of the initial source S_c are scattered in the system until they are lost by leakage or absorption. In case of fission they create the source distribution of the next cycle, and so on as shown in Fig. 1. This is the source iteration procedure. It is obtained by splitting the last Eqn. 10 into a system of coupled integral equations of the form:

$$\psi_i = S_c + (TC)\psi_i + S_i \quad (11)$$

which are all neutron transport equations for source problems for a nonmultiplying medium.

Here:

$$S_1 = S_c = TS\ ,$$

and:

$$S_i = (TF)\psi_i \qquad i = 2,3,4,\ \dots .$$

$$
\begin{aligned}
\psi_1 &= (TC)\psi_1 + S_c && \Rightarrow\ S_2 = TF\psi_1 \\
\psi_2 &= (TC)\psi_2 + TF\psi_1 && \Rightarrow\ S_3 = TF\psi_2 \\
\psi_3 &= (TC)\psi_3 + TF\psi_2 && \Rightarrow\ S_4 = TF\psi_3 \\
\psi_4 &= (TC)\psi_4 + TF\psi_3 && \Rightarrow\ S_5 = TF\psi_4 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\psi_{n-1} &= (TC)\psi_{n-1} + TF\psi_{n-1} && \Rightarrow\ S_n = TF\psi_{n-1} \\
\psi_n &= (TC)\psi_n + TF\psi_{n-1}
\end{aligned}
$$

$$
\begin{aligned}
\psi = \sum_{i=1}^{n}\psi_i &= (TC)\sum_{i=1}^{n}\psi_i + S_c + TF\sum_{i=1}^{n-1}\psi_i \\
&= (TC)\psi + S_c + TF\sum_{i=1}^{n}\psi_i - TF\psi_n \\
&= (TC+TF)\psi + S_c - TF\psi_n
\end{aligned}
$$

1. $As\ \ n \to \infty,\ \ TF\psi_n \to 0, subcritical\ \ reactor,$

   $\psi = (TC+TF)\psi + S_c, source\ problem.$

2. $If\ \psi_{n-1} \to \psi_n,\ as\ n \to \infty,\ \ just\ critical\ reactor,$

   $\psi = (TC+TF)\psi,\ eigenvalue\ \ problem.$

Figure 1. Source Iteration Method with Absorption Weighting.

This is actually carried out in Monte Carlo calculations by considering successive neutron generations. The generation of the ψ_i's is through the stored statistical weights:

$$
\frac{\nu_f \Sigma_f}{\Sigma_t} W_{n-1}, \tag{12}
$$

as discussed earlier.

The summation of all ψ_i's leads to:

$$
\sum_{i=1}^{n}\psi_i = (TC)\sum_{i=1}^{n}\psi_i + TF\sum_{i=1}^{n}\psi_i - TF\psi_n + S_c \tag{13}
$$

If the reactor is subcritical, the sum:

$$\sum_{i=1}^{n} \psi_i = \psi \tag{14}$$

becomes a solution for the collision density equation, and:

$$(TF)\psi_n \to 0,\ as\ n \to \infty. \tag{15}$$

If, however, ψ_{n-1} is asymptotic to $\psi_n\ \ as\ n \to \infty$, then ψ_n becomes a solution of the following homogeneous integral equation which describes a just critical system:

$$\psi_n = (TC)\psi_n + (TF)\psi_n \tag{16}$$

The procedure allows us to calculate characteristic reactor parameters. For instance the ratio:

$$k_i = \frac{\int_R S_{i+1}(r)dr}{\int_R S_i(r)dr} \tag{17}$$

is the multiplication factor between neutron generations (I+1) and (i).

When, after some iterations, the fundamental mode is reached, then $k_{i-1}=k_i$, $\psi_i = k_i \psi_{i-1}$, and we can obtain the effective multiplication factor of stationary criticality as::

$$k_{effective}(stat) = \underset{i \to \infty}{Lim}(k_i)\ . \tag{18}$$

In $k_{eff}<1$, the series for ψ converges, and if $k_{eff}>1$, the procedure diverges since the system is supercritical, but the method can also handle such problems if an artificial reduction parameter such a virtual absorption cross section is used.

The dynamical criticality can also be obtained as:

$$k_{effective}(dyn) = \frac{\int_R \sum_{i=1}^{\infty} S_{i+1}(r)dr}{\int_R \sum_{i=1}^{\infty} S_i(r)dr} \tag{19}$$

which is the average multiplication factor of all neutron generations.

While this procedure is well suited to criticality problems, it is not well adapted to systems dominated by the source S with value of k<1. Moreover, implementing Absorption weighting, one encounters many practical difficulties related to the control of the number of secondaries, their normalization, storage and sampling that are necessary for accounting for the multiplication portion from generation to generation. The previously discussed statistical weighting methods would be suitable in such cases.

EXERCISE

1. By multiplying the weight adjustments with their associated probabilities, prove that the different statistical weighting methods discussed yield the same expected values for the initial particles and its associated secondaries.