ELLIPTICAL PARTIAL DIFFERENTIAL EQUATIONS WITH INTERNAL SOURCES

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INTRODUCTION

Partial differential equations (PDEs) occur in a wide variety of areas of interest in science and engineering. We consider the following second order linear partial differential equation:

$$A(x,y)\frac{\partial^2 u}{\partial x^2}+B(x,y)\frac{\partial^2 u}{\partial x \partial y}+C(x,y)\frac{\partial^2 u}{\partial y^2}=f(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y})\,. \qquad (1)$$

When f = 0, Eqn.1 possesses three standard canonical forms obtainable by a suitable change of variables. Partial differential equations can be classified according to the magnitudes of A, B, and C in Eqn. 1:

$$\begin{aligned} B^2-4AC &< 0 \Rightarrow Elliptic\ \ PDE \\ B^2-4AC &= 0 \Rightarrow Parabolic\ PDE \\ B^2-4AC &> 0 \Rightarrow Hyperbolic\ PDE \end{aligned} \qquad (2)$$

A special case of Eqn. 1 in the Elliptic category is the Poisson's equation with a mixed Dirichlet and Neumann type of boundary conditions:

$$\begin{aligned} &k\nabla^2 u+q=0\ \ on\ \ D, \\ &u=(af+b\frac{df}{dr})\ \ on\ C, \end{aligned} \qquad (3)$$

where: f is some prescribed function,
$\frac{df}{dr}$ is its derivative,
a and b are constants,
∇^2 is the Laplacian operator.

For a two dimensional situation:

$$k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+q=0 \qquad (4)$$

This equation can represent the steady state heat conduction equation with a conductivity k and an internal heat generation volumetric source q.

A random walk that is generated starting from an inner point and moving to the boundary, which is continually incremented by the value of the source at each collision point, determines the solution at the starting point. One can visualize the random walk as a stepwise trip undertaken by a inebriated person walking in a city, heading at each block intersection north, south, east or west randomly with equal probabilities, adding to his drinking at each intersection. His walk stops when he reaches the city wall or boundary, or continues his walk by being reflected back into the city, depending on whether the walls are absorbing or reflecting.

ANALYTICAL SOLUTION BENCHMARK

To test the proposed Monte Carlo procedure, we suggest a simple benchmark for which an exact analytical solution can be easily derived. We build the Monte Carlo model that can solve this analytical solution. With our acquired confidence, we can then apply our procedure to problems of mixed boundary conditions that neither possess analytical or numerical solutions nor experimental results.

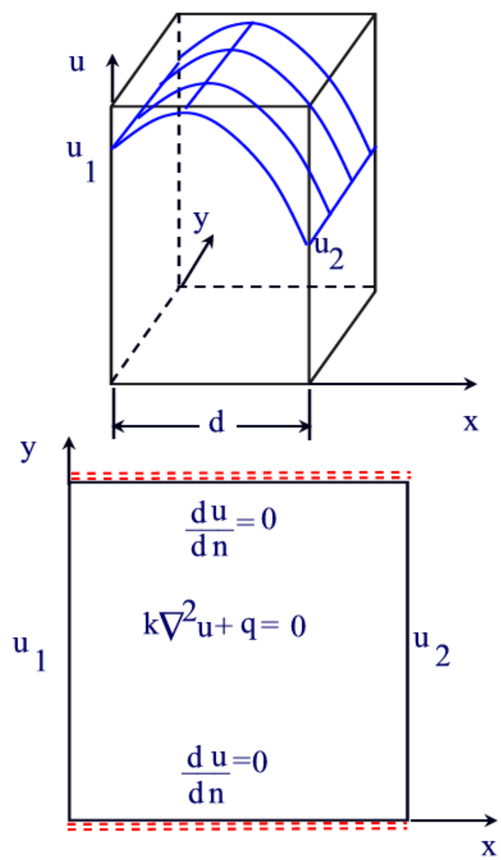


Figure 1. Two-dimensional Poisson's problem with mixed Dirichlet and Neumann boundary conditions.

We consider the two-dimensional square plate of side length d shown in Fig. 1. A set of boundary conditions of the mixed type are chosen in the sense that some boundaries have Dirichlet type conditions, whilst other boundaries have Neumann boundary conditions specified on them. In particular, let the left and right boundaries have Dirichlet boundary conditions, while the top and bottom boundaries have Neumann boundary conditions.

Practically, this problem could represent a plate heated at the left hand side to a temperature u_1, while the right hand side boundary is cooled and maintained at a temperature u_2. The top and bottom sides are insulated and constitute adiabatic insulated boundaries with no heat flux flowing through them. The plate would have a thermal conductivity k and a volumetric internal heat source q. The heat source is taken here as constant, but this is no limitation to the Monte Carlo procedure, as is can be taken as any distribution.

Alternatively, the square plate could represent an electrical conductor maintained at a voltage u_1 on the left side and grounded on the right hand side to a voltage u_2, with no current flow across the top and bottom boundaries, which are insulated. The electrical conductivity would still be k, and the space charge distribution would be q.

For a isotropic and homogeneous medium with uniform properties, the underlying governing equation is the Poisson's Equation. Because of the problem's symmetry along the y-axis, with the derivative to the solution zero everywhere, the solution along the y-axis is a constant everywhere and the equation needs only to be solved along the x-axis. Consequently we can write:

$$\begin{aligned} & k\nabla^2 u + q = 0, \\ & \frac{d^2u}{dx^2} = \frac{d}{dx}\left(\frac{du}{dx}\right) = -\frac{q}{k} \end{aligned} \tag{5}$$

Integrating once:

$$\int d\left(\frac{du}{dx}\right) = -\frac{q}{k}\int dx \,, \Rightarrow \frac{du}{dx} = -\frac{qx}{k} + a \tag{6}$$

where: a is an integration constant.

Integrating a second time, we get:

$$\int du = \int \left(-\frac{qx}{k} + a\right) dx \,, \Rightarrow u(x) = -\frac{qx^2}{2k} + ax + b \tag{7}$$

Applying the two boundary conditions:

$$\begin{aligned} & u = u_1 \;\; for \;\; x = 0 \,, \\ & u = u_2 \;\; for \;\; x = d \,. \end{aligned} \tag{8}$$

yields:

$$b = u_1 \ ,$$
$$a = \frac{(u_2 - u_1)}{d} + \frac{qd}{2k} \qquad (9)$$

Substituting from Eqn. 9 into Eqn. 7 we get:

$$u(x) = u_1 - \ \frac{(u_1 - u_2)}{d} \ x + \frac{qd}{2k} x + \frac{q}{2k} x^2 \qquad (10)$$

which is the equation of a parabola intersecting the y axis at u_1.

As a result of the presence of the volumetric internal source q, u(x) may reach a maximum whose location can be determined if we set the first derivative equal to zero:

$$\frac{du(x)}{dx} = \frac{(u_1 - u_{2)}}{d} \ + \frac{qd}{2k} + \frac{q}{2k} x = 0 \qquad (11)$$

Solving for the position of the maximum, we get:

$$x_{\max} = \frac{d}{2} - (u_1 - u_2) \frac{k}{qd} \qquad (12)$$

The last Eqn. 12 suggests that the position of the maximum would occur at the left of the midpoint if $u_1 > u_2$ and at its right otherwise. It also depends on the conductivity k, the volumetric source strength q, and the dimension d. The position of the maximum would occur at the center d/2, if $u_1 = u_2$.

A procedure that generates the exact analytical value for the benchmark solution for different values of the parameters is shown in Fig. 2. The graph generated by the analytical result for d = 1.0, k = 0.1, and q = 50, is shown in Fig. 3 for testing and comparison to the results obtained by the Monte Carlo procedure.

```
! Program Poisson_Analytical
! Calculates analytical solution for Poisson's benchmark
! M. Ragheb
        program Poisson_Analytical
        dimension u(100),x(100)
        real u,dx
        real q0,k,d,u1,u2,x,xx,z
        integer n
! Axial dimension
        d=1.0
! Number of nodes
        n=30
! Node spacing
        dx=1.0/n
        x(1)=0.0
        do i=2,n+1
                x(i)=(i-1)*dx
        end do
! Volumetric source
```

```
        q0=50.0
! Conductivity
        k=0.1
! Left boundary condition
        u1=100.
! Right boundary condition
        u2=0.0
! Open output file for visualization in Excel
        open (unit=10,file='profile.xls',status='unknown')
! Calculate solution at different nodes
! Left boundary
        z=x(1)
        call sol(z,xx,q0,k,d,u1,u2)
        u(1)=xx
        write(10,*) u(1),x(1)
        write(*,*) u(1),x(1)
        do i=2,n+1
                z=x(i)
                call sol(z,xx,q0,k,d,u1,u2)
                u(i)=xx
                write(10,*) u(i),x(i)
                write(*,*) u(i),x(i)
        end do
        end
! Define solution
        subroutine sol(z,xx,q0,k,d,u1,u2)
        real q0,k,d,u1,u2,x,xx,z
        xx=-(q0/(2.0*k))*z*z+(q0*d/(2.0*k))*z+((u2-u1)/d)*z+u1
        return
        end
```

Figure 2. Procedure to generate the exact analytical result for the benchmark problem.

RANDOM WALK MODEL

We consider the finite difference approximation of the ∇^2 operator in the x direction:

$$\frac{d^2u}{dx^2} = \frac{d}{dx}\left(\frac{du}{dx}\right) = \frac{\left(\frac{\Delta u}{\Delta x}\right)_+ - \left(\frac{\Delta u}{\Delta x}\right)_-}{\Delta x} = 0 \tag{13}$$

From which:

$$\frac{\left(u_{i+1,j} - u_{i,j}\right) - \left(u_{i,j} - u_{i-1,j}\right)}{\left(\Delta x\right)^2} = 0 \tag{14}$$

Consequently, in the x direction:

$$\frac{\left(u_{i+1,j} + u_{i-1,j} - 2u_{i,j}\right)}{\left(\Delta x\right)^2} = 0 \tag{15}$$

Similarly, in the y direction:

$$\frac{d^2u}{dy^2} = \frac{d}{dy}\left(\frac{du}{dy}\right) = \frac{\left(\frac{\Delta u}{\Delta y}\right)_+ - \left(\frac{\Delta u}{\Delta y}\right)_-}{\Delta y} = 0 \qquad (16)$$

$$\frac{(u_{i,j+1} - u_{i,j}) - (u_{i,j} - u_{i,j-1})}{(\Delta y)^2} = 0 \qquad (17)$$

$$\frac{(u_{i,j+1} + u_{i,j-1} - 2u_{i,j})}{(\Delta y)^2} = 0 \qquad (18)$$

Let us choose:

$$\Delta x \equiv \Delta y \equiv h,$$

thus we can rewrite Eqns. 15 and 18 as:

$$\frac{d^2u}{dx^2} \approx \frac{(u_{i+1,j} + u_{i-1,j} - 2u_{i\,j})}{h^2} = 0. \qquad (19)$$

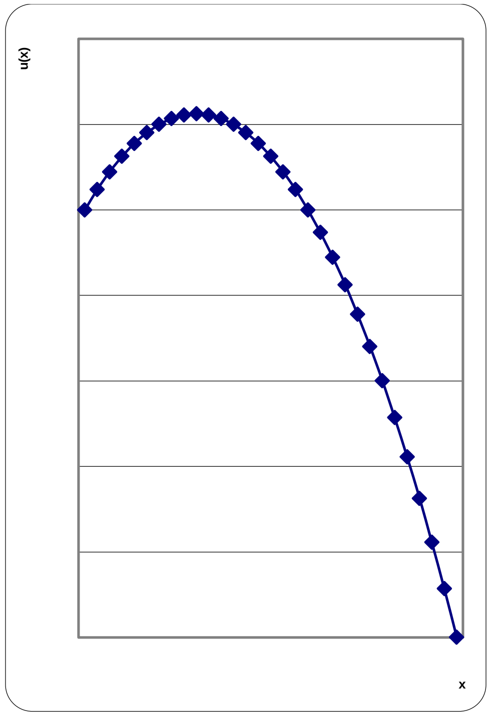
$$\frac{d^2u}{dy^2} \approx \frac{(u_{i,j+1} + u_{i,j-1} - 2u_{i,j})}{h^2} = 0 \qquad (20)$$

In two-dimensional problems we can add Eqns. 19 and 20 to yield:

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i\,j}}{h^2} \qquad (21)$$

Thus we write the finite difference form of Poisson's Eqn. 4 as:

$$k\left[\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2}\right] + q = k\left(\frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i\,j}}{h^2}\right) + q = 0 \qquad (22)$$



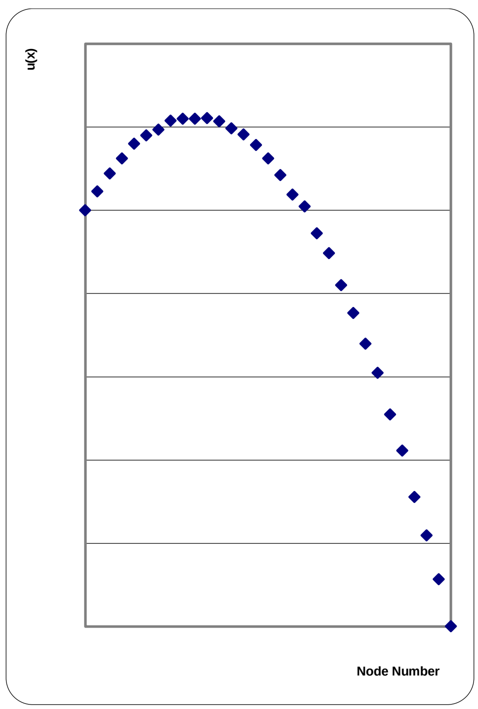


Figure 3. Comparison of benchmark's exact analytical results (top), to Monte Carlo Procedure's results (bottom) for N=10,000.

Solving for $u_{i,j}$, we get:

$$u_{i,j} = \frac{(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1})}{4} + \frac{h^2 q}{4k} \tag{23}$$

This implies that each internal point is an average over the surrounding node points in addition to a contribution from the internal source equal to:

$$\Delta s = \frac{1}{4}\left(\frac{h^2 q}{k}\right) \tag{24}$$

The averaging process is a characteristic of potential problems in general. This implies a random with equal probabilities of moving from a mesh point to its four immediate neighbors in two dimensions, and to its six immediate neighbors in three dimensions with an internal source contribution also averaged over the immediate neighbors as given by Eqn. 24.

MONTE CARLO PROCEDURE

A procedure was developed for the solution of the Poisson's equation with a mixed type of Dirichlet and Neumann boundary conditions. The boundary value, whenever the random walk reaches the boundary is added to the estimate of the solution. This is replaced by a reflection of the particle back to its original position whenever it reaches a reflecting boundary. The internal source contributes a value at each collision point.

The solution is stored either as an Excel file or an Array Visualizer file, depending on which visualization option will be used. For a given problem at hand, the boundary conditions can be modified accordingly. If the medium is anisotropic, the transition probabilities can be easily modified.

```
!	Poisson_solver_profile_mixed.for
	program Poisson_solver_profile_mixed
!	Two-dimensional Poisson's Equation Solver, with profile generation
!	Poisson's Equation in two Dimensions, Cartesian Coordinates.
!	Solution by Monte Carlo random walk on a rectangular surface
!	with mixed Dirichlet and Neumann boundary conditions
!	M. Ragheb
!	University of Illinois at Urbana-Champai
!	Random walk with equal step sizes
	dimension score(31,31), temp(31,31)
!	real(8) elapsed_time,k,source,s,delta,score,temp
	character*1 tab
	elapsed_time=timef()
	tab=char(9)
!	Volumetric heat source on domain
	s=50.0
!	Conductivity on domain
	k=0.1
!	k=1.0
!	Number of nodes
	xn=30
!	Node spacing dx=dy
	delta=1.0/xn
```

```
!       Incremental contribution from internal source
        source=(s*delta*delta)/(4.0*k)
!       Store output matrix for visualization using Excel
!       open (unit=10, file='temp_profile.xls', status='unknown')
!       Store output matrix for visualization using the Array visualizer
        open (unit=10, file='temp_profile.agl', status='unknown')
!       m1 = number of mesh points in x-direction
        m1=31
!       n1 = number of mesh points in y-direction
        n1=31
!       step probabilities in x+:p1, y+:p2, x-:p3 and y-:p4 directions
        p1=0.25
        p2=0.25
        p3=0.25
        p4=0.25
!       Construct cumulative distribution function for random walk
        p12=p1+p2
        p123=p1+p2+p3
!       number of random walks: nsamp
        nsamp=10000
!       Boundary conditions on the rectangle
!       left t1=t(0,y), bottom t2=t(x,0), right t3=t(m1,y), upper t4=t(x,n1)
        t1=100.
!       t2=0.
        t3=0.
!       t4=0.
!       Coordinates indices of point at which solution is estimated
!       m: i0=x/dx, n: j0=y/dy
        do 30 m=2,30,1
        do 30 n=2,30,1
!       Start random walk here
!       Initialize counters
!       History counter
        ncount=0
!       Score counter
        score(m,n)= source
!       Initiate random walk
99      i=m
        j=n
999     call random(r)
!       Sample cumulative distribution function for random walk
!       Move one step to the right
        if (r le.p1) then
        i=i+1
        score(m,n)=score(m,n)+source
        goto 11
!       Move one step up
        else if (r le.p12) then
        j=j+1
        score(m,n)=score(m,n)+source
        goto 11
!       Move one step left
        else if (r le.p123) then
        i=i-1
        score(m,n)=score(m,n)+source
        goto 11
        else
!       Move one step down
        j=j-1
        score(m,n)=score(m,n)+source
        goto 11
        end if
```

```
!          Check for random walk reaching boundary
!          Check whether lower boundary is reached
11         if (j.eq.1) then
!          score(m,n) = score(m,n) + t2
!                     Reflective boundary condition
           j=j+1
!          goto 88
!                     Continue random walk, do not terminate history
           goto 999
!          Check whether right boundary is reached
           else if (i.eq m1) then
           score(m,n) = score(m,n) + t3
           goto 88
!          Check whether upper boundary is reached
           else if (j.eq n1) then
!          score(m,n) = score(m,n) + t4
!                     Reflective boundary condition
           j=j-1
!          goto 88
!                     Continue random walk, do not terminate history
           goto 999
!          Check whether left boundary is reached
           else if (i.eq.1)  then
           score(m,n) = score(m,n) + t1
!          write(*,*) score
           goto 88
           else
           goto 999
           end if
!          Increment history counter
88         ncount=ncount+1
!          Check for total number of histories
           if (ncount lt nsamp) then
           goto 99
           else
           goto 77
           end if
!          Calculate solution at points of interest
77         xcount=ncount
           temp(m,n)=score(m,n)/xcount
!          write(10,*) temp(m,n)

!          write(*,*) score
!          write(*,*) xcount
!          Print results
30         continue
           write(*,*)'number of random walks=',nsamp
!          write(*,*)'coordinate of point at which temperature is calculated',m,n
!          write(*,*)'calculated temperature=',temp
!          Boundary values
           do 40 i=1,31
                      do 40 j=1,31
!                     bottom boundary
                      temp(i,1)=temp(i,2)
!                     top boundary
                      temp(i,31)=temp(i,30)
!                     left boundary
                      temp(1,j)=100.0
!                     right boundary
                      temp(31,j)=0.0
40         continue
           write(*,300) temp
```

```
          do 20 n=1,31
          write(10,300) (temp(m,n),tab, m=1,31)
20        continue
300       format(31(e14.8,a1))
!         elapsed_time=timef()
          write(*,*) elapsed_time
!
          stop
          end
```

Figure 4. Outline of Monte Carlo procedure for the solution of Poisson's equation with an internal source and mixed boundary conditions.

First, the procedure's adequacy is tested against the derived benchmark, for which we know the exact analytical solution. This is shown in Fig. 3, and was used to fine-tune the model until agreement between the exact solution and the Monte Carlo procedure was reached. Figure 4 shows the listing of the procedure displaying the addition of the internal source contribution at each collision point in the random walk. The full solution to the benchmark is shown in Fig. 5. A feature that is observable is the distribution reaching a peak caused by the presence of the internal volumetric source. Another interesting feature is the localized behavior of the solution, which appears so much similar to the analytical solution as a straight line close to the reflecting boundary. With this agreement between the analytical benchmark and the Monte Carlo model, we can then proceed to apply the procedure to a problem whose analytical, or even numerical solution would be hard to obtain otherwise.

TREATMENT OF ANISOTROPIC CONDUCTION

We consider the finite difference approximation of the anisotropic heat conduction with $k_x \neq k_y$. The Laplacian ∇^2 operator in the x direction takes the form:

$$\frac{d}{dx}\left(k_x \frac{du}{dx}\right) = \frac{\left(k_x \frac{\Delta u}{\Delta x}\right)_+ - \left(k_x \frac{\Delta u}{\Delta x}\right)_-}{\Delta x} = 0 \qquad (25)$$

From which:

$$\frac{k_x\left(u_{i+1,j} - u_{i,j}\right) - k_x\left(u_{i,j} - u_{i-1,j}\right)}{\left(\Delta x\right)^2} = 0 \qquad (26)$$

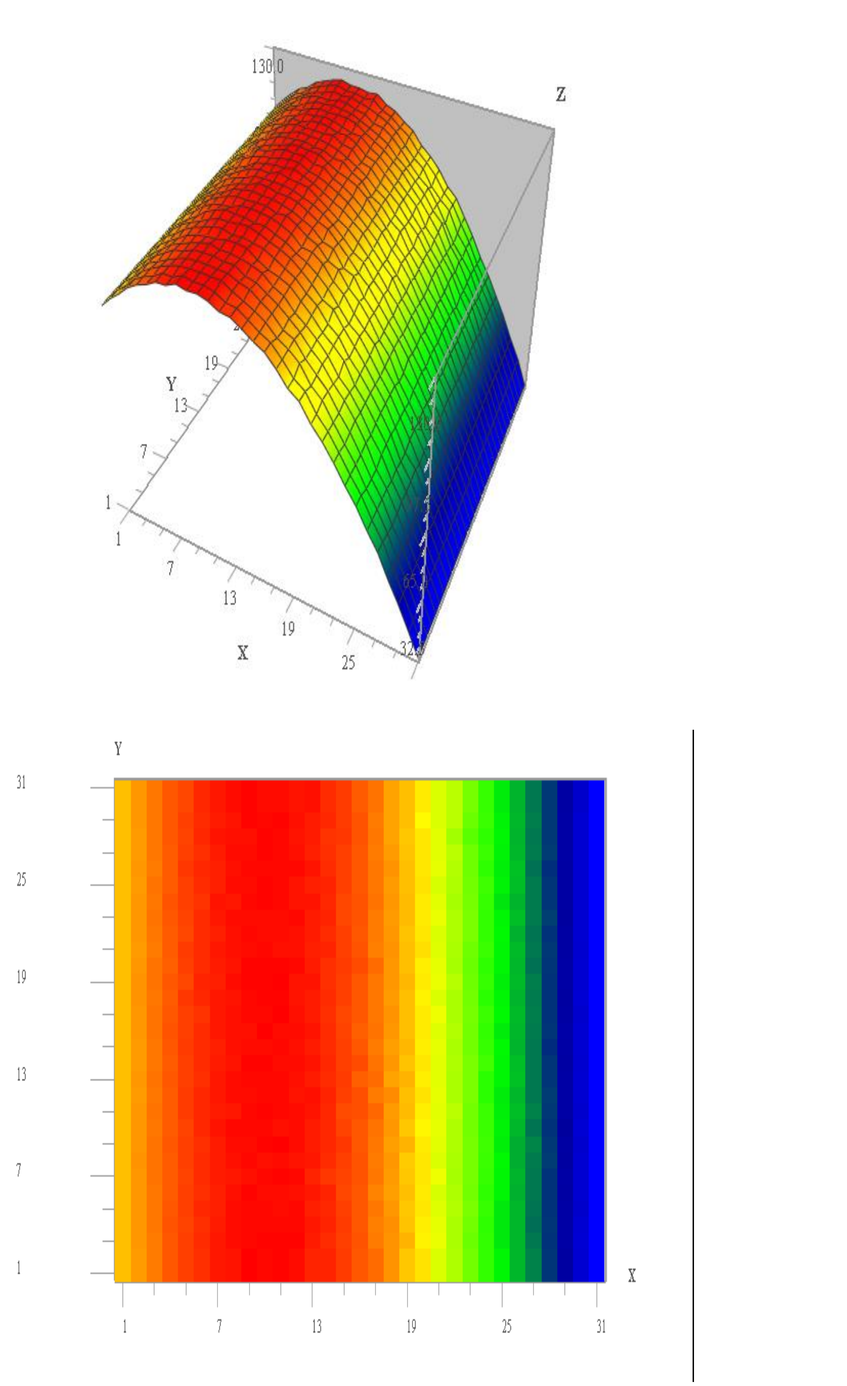


Figure 5. Solution of Poisson’s problem, $k = 0.1$, $q_0 = 50.0$, $N = 10{,}000$.

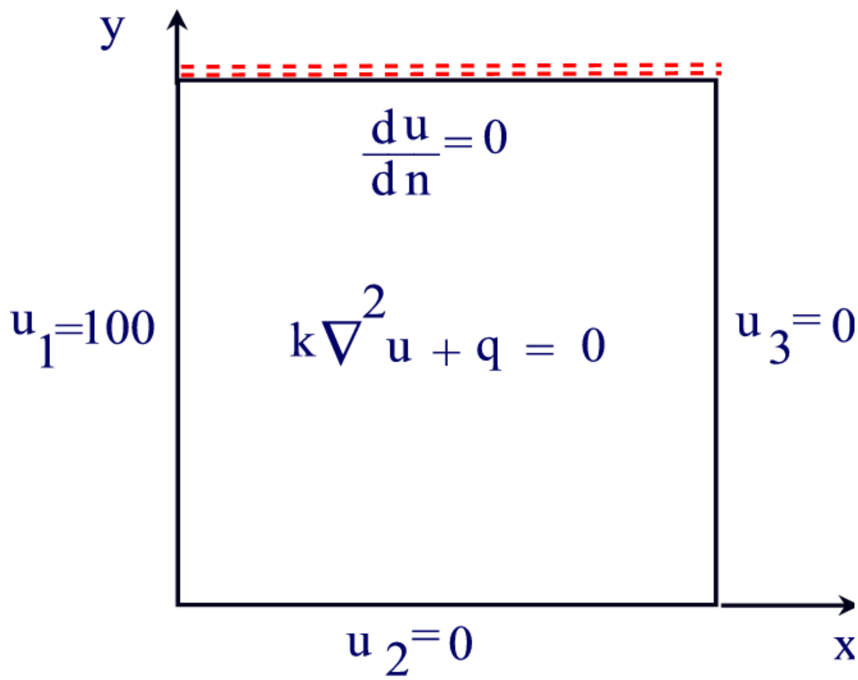


Figure 6. Geometry and boundary conditions for Poisson's mixed type Dirichlet and Neumann boundary conditions.

Consequently, in the x direction:

$$\frac{k_x\left(u_{i+1,j}+u_{i-1,j}-2u_{i,j}\right)}{(\Delta x)^2}=0 \tag{27}$$

Similarly, in the y direction:

$$\frac{d}{dy}\left(k_y\frac{du}{dy}\right)=\frac{k_y\left(\frac{\Delta u}{\Delta y}\right)_+-k_y\left(\frac{\Delta u}{\Delta y}\right)_-}{\Delta y}=0 \tag{28}$$

$$\frac{k_y\left(u_{i,j+1}-u_{i,j}\right)-k_y\left(u_{i,j}-u_{i,j-1}\right)}{(\Delta y)^2}=0 \tag{29}$$

$$\frac{k_y\left(u_{i,j+1}+u_{i,j+1}-2u_{i,j}\right)}{(\Delta y)^2}=0 \tag{30}$$

Let us choose:

$$\Delta x \equiv \Delta y \equiv h,$$

thus we can rewrite Eqns. 29 and 30 as:

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial u}{\partial x}\right) \approx \frac{k_x (u_{i+1,j} + u_{i-1,j} - 2u_{i\,j})}{h^2} = 0 \qquad (31)$$

$$\frac{\partial}{\partial y}\left(k_y \frac{\partial u}{\partial y}\right) \approx \frac{k_y \left(u_{i,j+1} + u_{i,j-1} - 2u_{i,j}\right)}{h^2} = 0 \qquad (32)$$

In two-dimensional problems we can add Eqns. 31 and 32 to yield:

$$\begin{aligned} &\frac{\partial}{\partial x}\left(k_x \frac{\partial u(x,y)}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial u(x,y)}{\partial y}\right) = \\ &\frac{k_x u_{i+1,j} + k_x u_{i-1,j} + k_y u_{i,j+1} + k_y\, u_{i,j-1} - 2(k_x + k_y) u_{i\,j}}{h^2} \end{aligned} \qquad (33)$$

Thus we write the finite difference form of the anisotropic conduction Poisson's equation as:

$$\begin{aligned} &\frac{\partial}{\partial x}\left(k_x \frac{\partial u(x,y)}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial u(x,y)}{\partial y}\right) + q = \\ &\frac{k_x u_{i+1,j} + k_x u_{i-1,j} + k_y u_{i,j+1} + k_y\, u_{i,j-1} - 2(k_x + k_y) u_{i\,j}}{h^2} + q = 0 \end{aligned} \qquad (34)$$

Solving for $u_{i,j}$, we get:

$$u_{i,j} = \frac{1}{2(k_x + k_y)}(k_x u_{i+1,j} + k_x u_{i-1,j} + k_y u_{i,j+1} + k_y\, u_{i,j-1}) + \frac{h^2 q}{2(k_x + k_y)} \qquad (35)$$

This implies that each internal point is a weighted average depending on the conductivity values in the different directions over the surrounding node points in addition to a contribution from the volumetric internal source equal to:

$$\Delta s = \frac{1}{2}\left(\frac{h^2 q}{k_x + k_y}\right) \qquad (36)$$

As a check on our derivation, Eqn. 36 for the anisotropic case reduces to Eqn. 24 for the isotropic case when $k_x = k_y = k$.

The averaging process is a characteristic of potential problems in general. This implies a random with equal probabilities of moving from a mesh point to its four immediate neighbors in two dimensions, and to its six immediate neighbors in three dimensions with an internal source contribution also averaged over the immediate neighbors given by Eqn. 35.

EXAMPLE

In case of a isotropic-conduction medium $k_x = k_y = k$, Eqns. 35 and 36, reduce into Eqns. 23 and 24, as expected. For other situations, the transfer probabilities in the random walk procedure would have to be modified accordingly. For instance, for double the conductivity in the x direction than in the y direction:

$$k_x = 2k_y,$$

Eqn. 35 becomes:

$$u_{i,j} = (\frac{1}{3}u_{i+1,j} + \frac{1}{3}u_{i-1,j} + \frac{1}{6}u_{i,j+1} + \frac{1}{6}u_{i,j-1}) + \frac{h^2 q}{6k_y}$$

In this case the transition probabilities in the random walk are equal to 1/6 in the north and south directions, and are equal to 1/3 in the east and west directions. Notice that the transition probabilities sum up to unity as should be the case for a probability density function (pdf).

POISSON MIXED DIRICHLET AND NEUMANN BOUNDARY VALUE PROBLEM

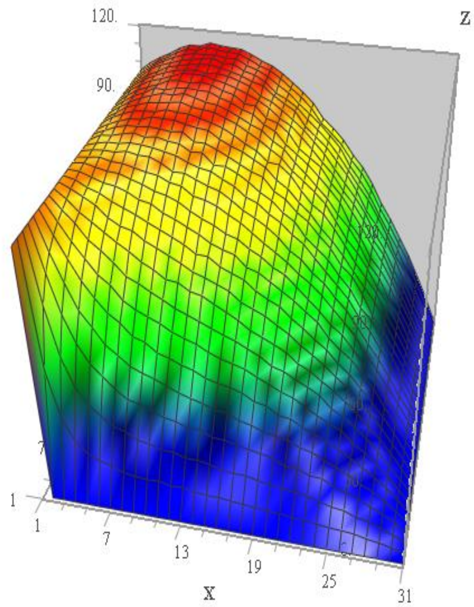
Having acquired confidence in the efficacy of the derived Monte Carlo procedure for solving such a class of problems, we are now able to apply it to a problem for which deriving an analytical solution would not be a straight forward task.

The geometry of the problem is shown in Fig. 6, and the Monte Carlo simulation result for N = 10,000 random walks generated per node is shown in Fig. 7.

The problem represents a square domain with a potential, temperature, or hydrostatic head applied to the left boundary, with a heat sink, electrical ground or fluid sink at the right and bottom boundaries. The top boundary represents a heat or electrically insulated or an impervious medium boundary medium. This otherwise heat transport, electrical potential or hydrostatic problem is mathematically speaking a mixed Dirichlet and Neumann boundary condition problem. In addition, an internal volumetric source q, of constant magnitude exists within the whole domain.

The result from the Monte Carlo random walk procedure in Fig. 7 shows that the boundary conditions are well satisfied close to the boundary. It is also remarkable to notice the local influence of the boundary conditions on the result. Close to the insulated top boundary, the result appears as a parabola as resulted close to the reflecting boundary in the benchmark problem. The parabolic shape smoothly turned into a curved solution that decreases from the left boundary representing the source to the right and bottom boundaries representing the sink.

Figure 8 also adds the treatment of a medium with anisotropic heat conduction, with the conductivity in the x direction as twice that in the y direction. Comparing with Fig. 8 for an isotropic conduction medium, a lower peak in the temperature distribution can be noticed as a result of the increased conductivity in the x direction.



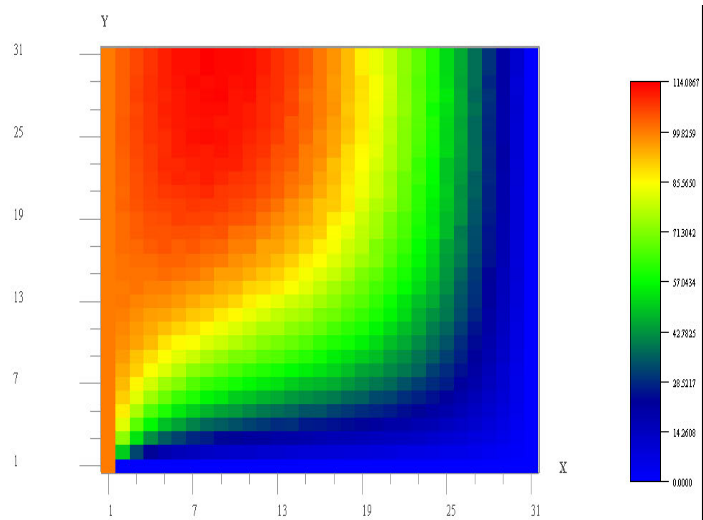


Figure 7. Poisson's boundary value problem, with volumetric source, $q_0 = 50$, $k = 0.1$, $N = 10{,}000$.

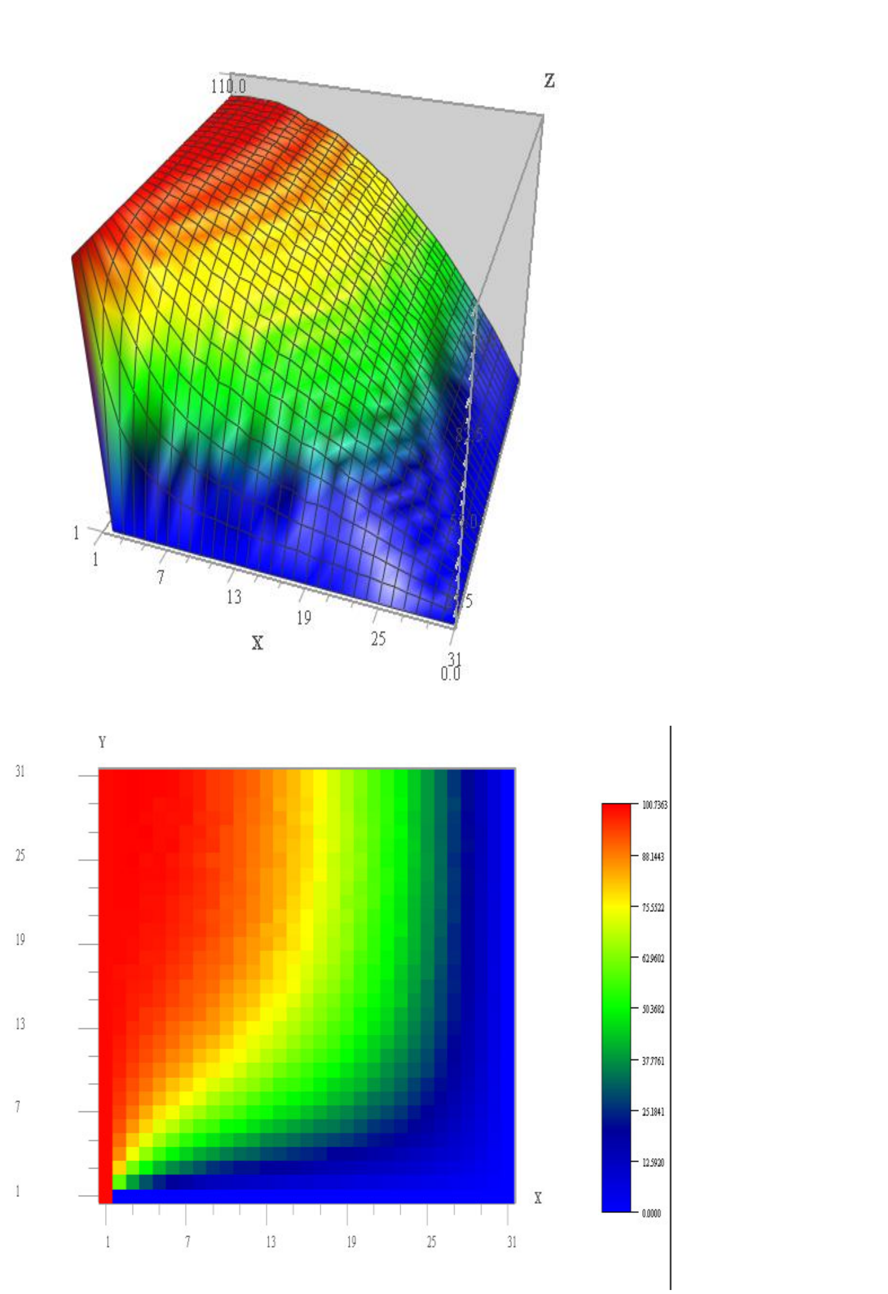


Figure 8. Poisson's boundary value problem, with volumetric source, q0=50, k_y=0.1, anisotropic conduction medium, $k_x = 2k_y$, N=10,000.

DISCUSSION

The Monte Carlo method offers a valuable tool for the treatment of the class of problems encountered in science and engineering designated as Elliptic partial differential equations. Here the Poisson's equation is considered where the boundary conditions are of the mixed Dirichlet and Neumann types. Analytical solutions could be hard to derive, and other numerical solutions may not be able to deal with non-linearity and higher dimensions. When Monte Carlo simulations are first undertaken, it is advisable to check the results of the generated algorithm against a benchmark of choice.

The Monte Carlo procedure, when first devised had to be fine-tuned to fit the results of the benchmark for the particular problem at hand. This approach is crucial to the development of an unbiased-solution model that represents the real problem at hand. The benchmark would normally solve a simpler problem for which an analytical solution can be derived. One could also use the solution generated by another numerical method. Best, the Monte Carlo simulation should be compared to any available experimental results.

EXERCISES

1. Repeat the mixed boundary value with internal source, comparing values of:
a) Conductivity, e.g. k = 0.1, 1.0.
b) Internal source, e.g. q = 50 and 5.
Comment on the results obtained.
2. Repeat the previous problem with a value of the internal source taking a non-constant value over a square geometry of side length d. For instance consider a sine distribution:

$$q(x,y) = q_0 \sin(\frac{\pi x}{d}).\sin(\frac{\pi y}{d}),$$

which would emulate the heat generation source distribution in a fission reactor.
3. By taking the boundary conditions at the right and left boundaries:

$$u_1 = u_2,$$

check the prediction of the analytical benchmark that the maximum of the distribution of the result would occur at the center of the square plate.