FLUID MECHANICS, EULER AND BERNOULLI EQUATIONS

4/1/2013

INTRODUCTION

The early part of the 18^{th} century saw the burgeoning of the field of theoretical fluid mechanics pioneered by Leonhard Euler and the father and son Johann and Daniel Bernoulli.

We introduce the equations of continuity and conservation of momentum of fluid flow, from which we derive the Euler and Bernoulli equations. The Bernoulli equation is the most famous equation in fluid mechanics. Its significance is that when the velocity increases in a fluid stream, the pressure decreases, and when the velocity decreases, the pressure increases.

The Bernoulli equation is applied to the airfoil of a wind machine rotor, defining the lift, drag and thrust coefficients and the pitching angle.

THE MASS CONSERVATION OR CONTINUITY EQUATION

The continuity equation of fluid mechanics expresses the notion that mass cannot be created nor destroyed or that mass is conserved. It relates the flow field variables at a point of the flow in terms of the fluid density and the fluid velocity vector, and is given by:

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \overline{V}) = 0 \tag{1}$$

We consider the vector identity resembling the chain rule of differentiation:

$$\nabla.(\rho \overline{V}) \equiv \rho \nabla.\overline{V} + \overline{V}.\nabla \rho \tag{2}$$

where the divergence operator is noted to act on a vector quantity, and the gradient operator acts on a scalar quantity.

This allows us to rewrite the continuity equation as:

$$\frac{\partial \rho}{\partial t} + \overline{V}.\nabla \rho + \rho \nabla.\overline{V} = 0 \tag{3}$$

SUBSTANTIAL DERIVATIVE

We can use the substantial derivative:

$$\frac{D}{Dt} \equiv \underbrace{\frac{\partial}{\partial t}}_{\substack{Local\\Derivative}} + \underbrace{(\overline{V}.\nabla)}_{\substack{Convetive\\Derivative}} \tag{4}$$

where the partial time derivative is called the local derivative and the dot product term is called the convective derivative.

In terms of the substantial derivative the continuity equation can be expressed as::

$$\frac{D\rho}{Dt} + \rho\nabla.\overline{V} = 0 \tag{5}$$

MOMENTUM CONSERVATION OR EQUATION OF MOTION

Newton's second law is frequently written in terms of an acceleration and a force vectors as:

$$\overline{F} = m\overline{a} \tag{6}$$

A more general form describes the force vector as the rate of change of the momentum vector as:

$$\overline{F} = \frac{d}{dt}(m\overline{V}) \tag{7}$$

Its general form is written in term of volume integrals and a surface integral over an arbitrary control volume v as:

$$\oiiint_{\mathrm{v}} \frac{\partial(\rho\overline{V})}{\partial t} d\mathrm{v} + \oiint_{S} (\rho\overline{V}.dS)\overline{V} = -\oiiint_{\mathrm{v}} \nabla p d\mathrm{v} + \oiiint_{\mathrm{v}} \rho \overline{f} d\mathrm{v} + \oiiint_{\mathrm{v}} \overline{F}_{viscous} d\mathrm{v} \tag{8}$$

where the velocity vector is:

$$\overline{V} = u\hat{x} + v\hat{y} + w\hat{z} \tag{9}$$

The cartesian coordinates x, y and z components of the continuity equation are:

$$\begin{aligned}
&\oiiint_{\mathrm{v}} \frac{\partial(\rho u)}{\partial t} d\mathrm{v} + \oiint_{S} (\rho\overline{V}.dS)u = -\oiiint_{\mathrm{v}} \frac{\partial p}{\partial x} d\mathrm{v} + \oiiint_{\mathrm{v}} \rho f_x d\mathrm{v} + \oiiint_{\mathrm{v}} (F_x)_{viscous} d\mathrm{v} \\
&\oiiint_{\mathrm{v}} \frac{\partial(\rho v)}{\partial t} d\mathrm{v} + \oiint_{S} (\rho\overline{V}.dS)v = -\oiiint_{\mathrm{v}} \frac{\partial p}{\partial y} d\mathrm{v} + \oiiint_{v} \rho f_y d\mathrm{v} + \oiiint_{\mathrm{v}} (F_y)_{viscous} d\mathrm{v} \\
&\oiiint_{\mathrm{v}} \frac{\partial(\rho w)}{\partial t} d\mathrm{v} + \oiint_{S} (\rho\overline{V}.dS)w = -\oiiint_{\mathrm{v}} \frac{\partial p}{\partial z} d\mathrm{v} + \oiiint_{v} \rho f_z d\mathrm{v} + \oiiint_{\mathrm{v}} (F_z)_{viscous} d\mathrm{v}
\end{aligned} \tag{10}$$

In this equation the product:

$$(\rho \bar{V}.dS) \tag{11}$$

is a scalar and has no components.

NAVIER STOKES EQUATIONS

By using the divergence or Gauss's theorem the surface integral can be turned into a volume integral:

$$\begin{aligned}
\oiint_S (\rho \bar{V}.dS)u &= \oiint_S (\rho u \bar{V}).dS = \oiiint_{\text{v}} \nabla.(\rho u \bar{V})d\text{v} \\
\oiint_S (\rho \bar{V}.dS)v &= \oiint_S (\rho v \bar{V}).dS = \oiiint_{\text{v}} \nabla.(\rho v \bar{V})d\text{v} \\
\oiint_S (\rho \bar{V}.dS)w &= \oiint_S (\rho w \bar{V}).dS = \oiiint_{\text{v}} \nabla.(\rho w \bar{V})d\text{v}
\end{aligned} \tag{12}$$

The volume integrals over an arbitrary volume now yield:

$$\begin{aligned}
\frac{\partial(\rho u)}{\partial t} + \nabla.(\rho u \bar{V}) &= -\frac{\partial p}{\partial x} + \rho f_x + (F_x)_{viscous} \\
\frac{\partial(\rho v)}{\partial t} + \nabla.(\rho v \bar{V}) &= -\frac{\partial p}{\partial y} + \rho f_y + (F_y)_{viscous} \\
\frac{\partial(\rho w)}{\partial t} + \nabla.(\rho w \bar{V}) &= -\frac{\partial p}{\partial z} + \rho f_z + (F_z)_{viscous}
\end{aligned} \tag{13}$$

These are known as the Navier-Stokes equations. They apply to the unsteady, three dimensional flow of any fluid, compressible or incompressible, viscous or inviscid.

In terms of the substantial derivative, the Navier-Stokes equations can be expressed as:

$$\begin{aligned}
\rho \frac{Du}{Dt} &= -\frac{\partial p}{\partial x} + \rho f_x + (F_x)_{viscous} \\
\rho \frac{Dv}{Dt} &= -\frac{\partial p}{\partial y} + \rho f_y + (F_y)_{viscous} \\
\rho \frac{Dw}{Dt} &= -\frac{\partial p}{\partial z} + \rho f_z + (F_z)_{viscous}
\end{aligned} \tag{14}$$

EULER EQUATIONS

For a steady state flow the time partial derivatives vanish. For inviscid flow the viscous terms are equal to zero. In the absence of body forces the f_x, f_y, anf f_z terms disappear. The Euler equations result as:

$$\begin{aligned}
\nabla.(\rho u\bar{V}) &= -\frac{\partial p}{\partial x} \\
\nabla.(\rho v\bar{V}) &= -\frac{\partial p}{\partial y} \\
\nabla.(\rho w\bar{V}) &= -\frac{\partial p}{\partial z}
\end{aligned} \tag{15}$$

INVISCID COMPRESSIBLE FLOW

For an inviscid flow without body forces, the momentum conservation equations of fluid mechanics are:

$$\begin{aligned}
\rho\frac{Du}{Dt} &= -\frac{\partial p}{\partial x} \\
\rho\frac{Dv}{Dt} &= -\frac{\partial p}{\partial y} \\
\rho\frac{Dw}{Dt} &= -\frac{\partial p}{\partial z}
\end{aligned} \tag{16}$$

These equations can also be written as:

$$\begin{aligned}
\frac{\partial(\rho u)}{\partial t} + \bar{V}.\nabla(\rho u) &= -\frac{\partial p}{\partial x} \\
\frac{\partial(\rho v)}{\partial t} + \bar{V}.\nabla(\rho v) &= -\frac{\partial p}{\partial y} \\
\frac{\partial(\rho w)}{\partial t} + \bar{V}.\nabla(\rho w) &= -\frac{\partial p}{\partial z}
\end{aligned} \tag{17}$$

For steady flow the partial time derivative vanishes, and we can write:

$$\begin{aligned}
\bar{V}.\nabla(\rho u) &= -\frac{\partial p}{\partial x} \\
\bar{V}.\nabla(\rho v) &= -\frac{\partial p}{\partial y} \\
\bar{V}.\nabla(\rho w) &= -\frac{\partial p}{\partial z}
\end{aligned} \tag{18}$$

Expanding the gradient term, we get:

$$\begin{aligned}
\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \rho w \frac{\partial u}{\partial z} &= -\frac{\partial p}{\partial x} \\
\rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} + \rho w \frac{\partial v}{\partial z} &= -\frac{\partial p}{\partial y} \qquad (19) \\
\rho u \frac{\partial w}{\partial x} + \rho v \frac{\partial w}{\partial y} + \rho w \frac{\partial w}{\partial z} &= -\frac{\partial p}{\partial z}
\end{aligned}$$

Rearranging, we get:

$$\begin{aligned}
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} \\
u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} &= -\frac{1}{\rho} \frac{\partial p}{\partial y} \qquad (20) \\
u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} &= -\frac{1}{\rho} \frac{\partial p}{\partial z}
\end{aligned}$$

STREAMLINES DIFFERENTIAL EQUATIONS

The definition of a streamline in a flow is that it is parallel to the velocity vector. Hence the cross product of the directed element of the streamline and the velocity vector is zero:

$$d\overline{s} \times \overline{V} = 0 \qquad (21)$$

where:

$$d\overline{s} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

$$\overline{V} = u\,\hat{x} + v\,\hat{y} + w\,\hat{z}$$

The cross product can be expanded in the form of a determinant as:

$$\begin{aligned}
d\overline{s} \times \overline{V} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ dx & dy & dz \\ u & v & w \end{vmatrix} \\
&= \hat{x}(w\,dy - v\,dz) + \hat{y}(u\,dz - w\,dx) + \hat{z}(v\,dx - u\,dy) \qquad (22) \\
&= 0
\end{aligned}$$

The vector being equal to zero, its components must be equal to zero yielding the differential equations for the streamline f(x,y,z) =0, as:

$$
\begin{aligned}
&w\,dy - v\,dz = 0 \\
&u\,dz - w\,dx = 0 \\
&v\,dx - u\,dy = 0
\end{aligned} \tag{23}
$$

EULER'S EQUATION

Multiplying the flow equations respectively by dx, dy and dz, we get:

$$
\begin{aligned}
&u\frac{\partial u}{\partial x}dx + v\frac{\partial u}{\partial y}dx + w\frac{\partial u}{\partial z}dx = -\frac{1}{\rho}\frac{\partial p}{\partial x}dx \\
&u\frac{\partial v}{\partial x}dy + v\frac{\partial v}{\partial y}dy + w\frac{\partial v}{\partial z}dy = -\frac{1}{\rho}\frac{\partial p}{\partial y}dy \\
&u\frac{\partial w}{\partial x}dz + v\frac{\partial w}{\partial y}dz + w\frac{\partial w}{\partial z}dz = -\frac{1}{\rho}\frac{\partial p}{\partial z}dz
\end{aligned} \tag{24}
$$

Using the streamline differential equations, we can write:

$$
\begin{aligned}
&u\frac{\partial u}{\partial x}dx + u\frac{\partial u}{\partial y}dy + w\frac{\partial u}{\partial z}dz = -\frac{1}{\rho}\frac{\partial p}{\partial x}dx \\
&u\frac{\partial v}{\partial x}dx + v\frac{\partial v}{\partial y}dy + w\frac{\partial v}{\partial z}dz = -\frac{1}{\rho}\frac{\partial p}{\partial y}dy \\
&u\frac{\partial w}{\partial x}dx + v\frac{\partial w}{\partial y}dy + w\frac{\partial w}{\partial z}dz = -\frac{1}{\rho}\frac{\partial p}{\partial z}dz
\end{aligned} \tag{25}
$$

The differentials of functions u = u(x,y,z), v = v(x,y,z), w =w (x,y,z) are:

$$
\begin{aligned}
&du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz \\
&dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz \\
&dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz
\end{aligned} \tag{26}
$$

This allows us to write:

$$
\begin{aligned}
udu &= -\frac{1}{\rho}\frac{\partial p}{\partial x}dx \\
vdv &= -\frac{1}{\rho}\frac{\partial p}{\partial y}dy \\
wdw &= -\frac{1}{\rho}\frac{\partial p}{\partial z}dz
\end{aligned} \tag{27}
$$

Through integration we can write:

$$
\begin{aligned}
\frac{1}{2}d(u^2) &= -\frac{1}{\rho}\frac{\partial p}{\partial x}dx \\
\frac{1}{2}d(v^2) &= -\frac{1}{\rho}\frac{\partial p}{\partial y}dy \\
\frac{1}{2}d(w^2) &= -\frac{1}{\rho}\frac{\partial p}{\partial z}dz
\end{aligned} \tag{28}
$$

Adding the three last equations we get:

$$
\begin{aligned}
\frac{1}{2}d(u^2+v^2+w^2) &= -\frac{1}{\rho}\left(\frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial y}dy+\frac{\partial p}{\partial z}dz\right) \\
\frac{1}{2}d(V^2) &= -\frac{1}{\rho}dp
\end{aligned} \tag{29}
$$

From the last equation we can write a simple form of Euler's equation as:

$$
dp = -\rho VdV \tag{30}
$$

Euler's equation applies to an inviscid flow with no body forces. It relates the change in velocity along a streamline dV to the change in pressure dp along the same streamline.

BERNOULLI EQUATION, INCOMPRESSIBLE FLOW

Considering the case of incompressible flow, we can use limit integration to yield:

$$
\begin{aligned}
&\int_{p_1}^{p_2} dp = -\rho\int_{V_1}^{V_2} VdV \\
&p_2 - p_1 = -\frac{\rho}{2}\left(V_2^2 - V_1^2\right) \\
&p_1 + \frac{1}{2}\rho V_1^2 = p_2 + \frac{1}{2}\rho V_2^2 = constant
\end{aligned} \tag{31}
$$

The relation between pressure and velocity in an inviscid incompressible flow was enunciated in the form of Bernoulli's equation, first presented by Euler:

$$p + \frac{1}{2}\rho V^2 = constant \tag{32}$$

This equation is the most famous equation in fluid mechanics. Its significance is that when the velocity increases, the pressure decreases, and when the velocity decreases, the pressure increases.

The dimensions of the terms in the equation are kinetic energy per unit volume. Even though it was derived from the momentum conservation equation, it is also a relation for the mechanical energy in an incompressible flow. It states that the work done on a fluid by the pressure forces is equal to the change of kinetic energy of the flow. In fact it can be derived from the energy conservation equation of fluid flow.

The fact that Bernoulli's equation can be interpreted as Newton's second law or an energy equation illustrates that the energy equation is redundant for the analysis of inviscid, incompressible flow.

ROTOR AIRFOIL GEOMETRY

The sharp end of an airfoil shape at point B is designated as the trailing edge. The leading edge is the locos of the point A of the nose of the airfoil profile that is the farthest from the trailing edge. The distance L = AB is known as the chord of the profile, with AMB being the upper surface and ANB the lower surface.

At any distance along the chord AB from the nose, points may be identified half way between the upper and lower surfaces whose locus is usually curved is called the camber line or median line of the airfoil section.

The incidence angle i is the angle between the chord and the air speed vector V at infinite upstream. The zero lift angle is the angle φ_0 between the chord and the zero lift line.

The lift angle is the angle φ between the zero lift line and the air speed vector V at infinite upstream. It is conventional to take φ and i as positive and φ_0 as negative. The following relations hold:

$$\begin{aligned} \varphi &= i + (-\varphi_0) = i - \varphi_0 \\ i &= \varphi - (-\varphi_0) = \varphi + \varphi_0 \end{aligned} \tag{33}$$

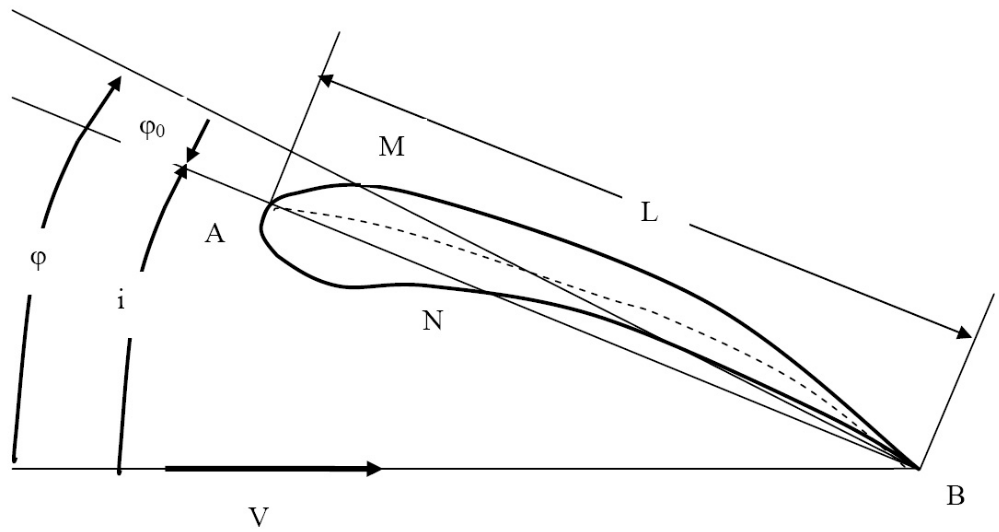

Fig.1: Rotor airfoil geometry.

Another rotor blade parameter is the maximum thickness h, which is sometimes expressed as a fraction of the chord length and is called the thickness/chord ratio or relative thickness.

The relative thickness can range from 3-20 percent, with the common wind machine rotors values covering the range of 10-15 percent.

The spot along the chord where the maximum thickness occurs in airfoils covers a range of 20 - 60 percent of the chord from the leading edge, and in wind machines rotors this is around 30 percent.

FORCES ON A MOVING ROTOR IN A STILL ATMOSPHERE

We can assume that the airfoil is at rest and the air moving at the same speed but in the opposite direction. In this case the aerodynamic force exerted on the rotor will not change in magnitude. The resultant force will depend only on the relative speed and the angle of attack. To simplify the analysis, the airfoil is taken at rest in a moving stream of air in an infinite upstream speed V.

The pressure of the air on the external surface of the airfoil will not be uniform due to the effect of the Bernoulli force. On the upper surface there results a lower pressure, and on the lower surface an increase in the pressure.

We can represent the pressure variation on the rotor surface by considering a line perpendicular to the airfoil profile surface whose magnitude is K_p , given by:

$$K_p = \frac{p - p_0}{\frac{1}{2}\rho V^2} \qquad (34)$$

where:

p *is the static pressure at the base of the perpendicular to the surface*

p_0 *is the pressure at the infinite upstream*

V *is the speed at the infinite upstream*

ρ *is the density at the infinite upstream*

The value of K_p is negative for the points at the upper surface of the airfoil, and positive for the lower surface.

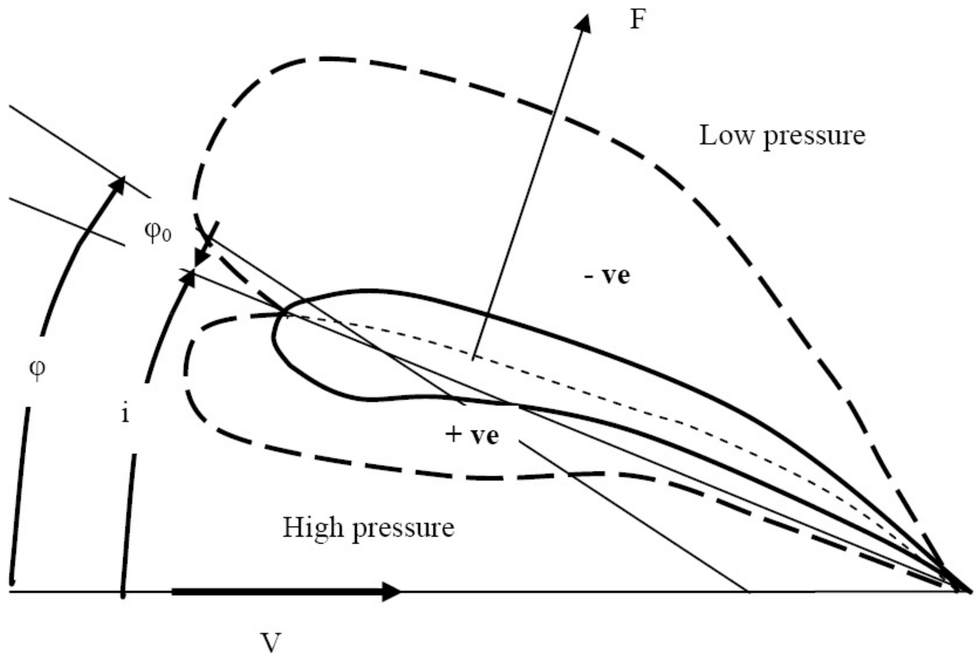

Fig. 2: Pressure profile on airfoil segment.

The resultant thrust force F_T is inclined with respect to the relative speed direction and is given by:

$$F_T = \frac{1}{2}\rho C_T A V^2 \tag{35}$$

where:

A *is the area equal to the product of the chord by the length of the rotor*

C_T *is the total aerodynamic coefficient*

The force F has two components. The first component is parallel to the velocity vector or the drag force:

$$F_D = \frac{1}{2}\rho C_D A V^2 \tag{36}$$

The second component is perpendicular to the velocity vector, or the lift force:

$$F_L = \frac{1}{2}\rho C_L A V^2 \tag{37}$$

These forces are perpendicular and we can apply the Pythagorean Theorem leading to the thrust force as:

$$F_T^2 = F_D^2 + F_L^2 \tag{38}$$

Consequently:

$$F_T = \sqrt{F_D^2 + F_L^2} \tag{39}$$

In addition:

$$\begin{gathered}\left(\frac{1}{2}\rho C_T A V^2\right)^2 = \left(\frac{1}{2}\rho C_L A V^2\right)^2 + \left(\frac{1}{2}\rho C_D A V^2\right)^2 \\ C_T^2 = C_D^2 + C_L^2\end{gathered} \tag{40}$$

where C_T is the thrust coefficient.

AERODYNAMIC MOMENT

If M is the aerodynamic moment of the force F relative to the leading edge, we can define a pitching moment coefficient C_m from the expression:

$$M = \frac{1}{2}\rho C_m A L V^2 \tag{41}$$

where:

L is the chord length.

The aerodynamic forces on the rotor may be represented by a lift, a drag, and a pitching moment.

At each value of the incidence angle there exists a particular point C about which the pitching moment of the aerodynamic force F is zero. This unique point is called the center of pressure. The aerodynamic effects on the airfoil section can be represented by the lift and the drag alone acting at that point.

The position of the center of pressure relative to the leading edge is calculated from the ratio:

$$CP = \frac{AC}{AB} = \frac{C_m}{C_L} \tag{42}$$

It is usually in the range of: 25 – 30 percent.

REFERENCES

1. Désiré Le Gouriérès, "Wind Power Plants, Theory and Design," Pergamon Press, 1982.

2. John D. Anderson, Jr., "Fundamentals of Aerodynamics," 3rd edition, McGrawHill, 2001.

EXERCISE

1. A wind rotor airfoil is placed in the air flow at sea level conditions with a free stream velocity of 10 m/s. The density at standard sea level conditions is 1.23 kg/m^3 and the pressure is 1.01 x 10^5 Newtons/m^2. At a point along the rotor airfoil the pressure is 0.90 x 10^5 Newtons/m^2. By applying Bernoulli's equation estimate the velocity at this point.

2. The lift force on a rotor blade is given by: $F_L = \frac{1}{2}\rho C_L A V^2$. The drag force is given by: $F_D = \frac{1}{2}\rho C_D A V^2$. Derive expressions for:

a) The thrust force F_T,

b) The thrust coefficient C_T.