

Vector Relations

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Element of Volume

Cartesian coordinates

$$dV_{car} = dx dy dz$$

Cylindrical coordinates

$$dV_{cyl} = r dr d\theta dz$$

Spherical coordinates

$$dV_{sph} = r^2 \sin \theta dr d\theta d\phi$$

Gradient

The gradient operates only on a scalar function ϕ and converts it into a vector. The gradient of a scalar function is the vector whose components are the rates of change of the function along the direction of the component.

Cartesian coordinates

$$grad_{car} \phi = \nabla_{car} \phi = \frac{\partial \phi}{\partial x} \hat{x} + \frac{\partial \phi}{\partial y} \hat{y} + \frac{\partial \phi}{\partial z} \hat{z}$$

Cylindrical coordinates

$$grad_{cyl} \phi = \nabla_{cyl} \phi = \frac{\partial \phi}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial \phi}{\partial \theta} \hat{\theta} + \frac{\partial \phi}{\partial z} \hat{z}$$

Spherical coordinates

$$grad_{sph} \phi = \nabla_{sph} \phi = \frac{\partial \phi}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial \phi}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \phi} \hat{\phi}$$

Divergence

Divergence Theorem

The divergence operates only on a vector \vec{F} and converts it into a scalar function. According to the divergence theorem, the integral of the normal component of a vector over a closed surface is equal to the integral of the divergence of the vector throughout the enclosed volume:

$$\iint_S \vec{F} \cdot \vec{n} \, dS = \iiint_V \operatorname{div} \vec{F} \, dV = \iiint_V \nabla \cdot \vec{F} \, dV,$$

where: \vec{n} is a unit vector normal to the surface S bounding the volume V.

Cartesian coordinates

$$\operatorname{div}_{car} \vec{F} = \nabla_{car} \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Cylindrical coordinates

$$\operatorname{div}_{cyl} \vec{F} = \nabla_{cyl} \cdot \vec{F} = \frac{1}{r} \frac{\partial}{\partial r} (r F_r) + \frac{1}{r} \frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}$$

Spherical coordinates

$$\operatorname{div}_{sph} \vec{F} = \nabla_{sph} \cdot \vec{F} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 F_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta F_\theta) + \frac{1}{r \sin \theta} \frac{\partial F_\phi}{\partial \phi}$$

Laplacian

This operator is the divergence of the gradient of a scalar function ϕ .

Cartesian coordinates

$$\nabla_{car}^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

Cylindrical coordinates

$$\nabla_{cyl}^2 \phi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2}$$

Spherical coordinates

$$\nabla_{sph}^2 \phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi}{\partial \phi^2}$$