

# Vector Relations

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## Element of Volume

**Cartesian coordinates**

$$dV_{car} = dx dy dz$$

**Cylindrical coordinates**

$$dV_{cyl} = r dr d\theta dz$$

**Spherical coordinates**

$$dV_{sph} = r^2 \sin \theta dr d\theta d\varphi$$

## Gradient

The gradient operates only on a scalar function  $\phi$  and converts it into a vector. The gradient of a scalar function is the vector whose components are the rates of change of the function along the direction of the component.

**Cartesian coordinates**

$$grad_{car} \phi = \nabla_{car} \phi = \frac{\partial \phi}{\partial x} \hat{x} + \frac{\partial \phi}{\partial y} \hat{y} + \frac{\partial \phi}{\partial z} \hat{z}$$

**Cylindrical coordinates**

$$grad_{cyl} \phi = \nabla_{cyl} \phi = \frac{\partial \phi}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial \phi}{\partial \theta} \hat{\theta} + \frac{\partial \phi}{\partial z} \hat{z}$$

**Spherical coordinates**

$$grad_{sph} \phi = \nabla_{sph} \phi = \frac{\partial \phi}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial \phi}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \varphi} \hat{\varphi}$$

## Divergence

**Divergence Theorem**

The divergence operates only on a vector  $\vec{F}$  and converts it into a scalar function. According to the divergence theorem, the integral of the normal component of a vector over a closed surface is equal to the integral of the divergence of the vector throughout the enclosed volume:

$$\iint_S \vec{F} \cdot \vec{n} dS = \iiint_V div \vec{F} dV = \iiint_V \nabla \cdot \vec{F} dV ,$$

where:  $\vec{n}$  is a unit vector normal to the surface S bounding the volume V.

**Cartesian coordinates**

$$\operatorname{div}_{car} \bar{F} = \nabla_{car} \cdot \bar{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

### Cylindrical coordinates

$$\operatorname{div}_{cyl} \bar{F} = \nabla_{cyl} \cdot \bar{F} = \frac{1}{r} \frac{\partial}{\partial r} (r F_r) + \frac{1}{r} \frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}$$

### Spherical coordinates

$$\operatorname{div}_{sph} \bar{F} = \nabla_{sph} \cdot \bar{F} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 F_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta F_\theta) + \frac{1}{r \sin \theta} \frac{\partial F_\varphi}{\partial \varphi}$$

## Laplacian

This operator is the divergence of the gradient of a scalar function  $\phi$ .

### Cartesian coordinates

$$\nabla_{car}^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

### Cylindrical coordinates

$$\nabla_{cyl}^2 \phi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2}$$

### Spherical coordinates

$$\nabla_{sph}^2 \phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi}{\partial \varphi^2}$$